A Method for Computing Split Weight Distributions of Reed-Muller Codes[1]

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Abstract — **This paper shows a method for computing the split weight distributions of Reed-Muller Code. By using this method, the split weight distribution of RM(3,9) is computed and presented. As a consequence, the weight distributions of the third and the fifth order Reed-Muller codes of length 512 are also computed.**

I. Introduction

The weight distribution of a linear block code can be computed if the split weight distribution of the code is known. For some codes, it is much easier to compute their split weight distributions, and then use the results to obtain the weight distributions of the codes. This paper presents a method for computing the split weight distribution of a Reed-Muller (RM) code. Using this method, the weight distributions of the third order and the fifth order RM codes of length 512 are computed.

II. Background

For linear block codes, A and B, such that $A \supseteq B$, let A/B denote the set of cosets of B in A, and let $[A/B]$ be the set of chosen coset leaders of A/B. For two binary sequences $\bar{u}$ and $\bar{v}$, let $\bar{u} \circ \bar{v}$ denote the concatenation of $\bar{u}$ and $\bar{v}$. Let $A \circ B$ be defined as $\{\bar{u} \circ \bar{v} : \bar{u} \in A \text{ and } \bar{v} \in B\}$. The following facts (Fact 1) to (Fact 4) are used for computing the split weight distributions of Reed-Muller codes.

(Fact 1) [Example 3 in [2, 3]] For linear block codes C_1 and C_2 such that $C_1 \supseteq C_2$, let C be a linear block code of length N constructed by $|\bar{u}|\bar{u}+\bar{v}|$ construction [1] from C_1 and C_2, that is, $C \stackrel{\triangle}{=} \{\bar{u} \circ (\bar{u}+\bar{v}) : \bar{u} \in C_1 \text{ and } \bar{v} \in C_2\}$. Then $C = \{(\bar{u}+\bar{v}_1) \circ (\bar{u}+\bar{v}_2) : \bar{u} \in [C_1/C_2], \bar{v}_1 \in C_2 \text{ and } \bar{v}_2 \in C_2\}$ and therefore $\{\bar{u} \circ \bar{u} : \bar{u} \in [C_1/C_2]\}$ can be chosen as $[C/(C_2 \circ C_2)]$. Let $\{B_1 = C_2, B_2, \cdots, B_s\}$ be a partition of C_1/C_2 such that, for each block B_i with $1 \le i \le s$, every coset in B_i has the same weight enumerator $W_i(x,y)$, and let ν_i denote the number of cosets in B_i. For a nonnegative integer $w \le N/2$, let $A_i(w)$ denote the number of codewords with weight w in a coset in B_i. Note that the weight enumerator $W_i(x,y)$ of the coset B_i is given by

$$W_i(x,y) = \sum_{h=0}^{N/2} A_i(h) x^{N/2-h} y^h.$$

For two integers w_L and w_R such that $0 \le w_L \le N/2$ and $0 \le w_R \le N/2$, let $A_{sp}(w_L, w_R)$ denote the number of codewords whose left and right weight [1] are w_L and w_R, respectively. Then, the split weight distribution of C is given by

$$A_{sp}(w_L, w_R) = \sum_{i=1}^{s} \nu_i A_i(w_L) A_i(w_R). \qquad (1)$$

Simultaneously, the weight enumerator $W(x,y)$ of C is given by

$$W(x,y) = \sum_{i=1}^{s} \nu_i W_i^2(x,y). \qquad (2)$$

(Fact 2) Let $RM(r,m)$ denote the r-th order Reed-Muller code of length 2^m. It is known [1] that $RM(r,m)$ can be constructed by the $|\bar{u}|\bar{u}+\bar{v}|$ construction from $RM(r,m-1)$ and $RM(r-1,m-1)$ with $0 < r < m$ and the general linear group $GL(m,2)$ permutes the codewords of $RM(r,m)$. Two cosets in $RM(r,m)/RM(r-1,m)$ are said to be $GL(m,2)$-*equivalent* if and only if one is a permutation of another by a transformation of $GL(m,2)$. Cosets in the same $GL(m,2)$-*equivalent* class, or orbit, have the same weight distribution. Let $s(r,m)$ denote the number of $GL(m,2)$-*orbits* in $RM(r,m)/RM(r-1,m)$ for $0 < r \le m$, and suppose that the $GL(m,2)$-*orbits* are numbered 1 to $s(r,m)$ and the first orbit consists of $RM(r-1,m)$, only. For $1 \le i \le s(r,m)$, let $\nu_i(r,m)$ denote the number of cosets in the i-th $GL(m,2)$-*orbit* in $RM(r,m)/RM(r-1,m)$ and let $W_{r,m,i}(x,y)$ and $a_{r,m,i}(w)$ denote the weight enumerator of a coset in the i-th orbit in $RM(r,m)/RM(r-1,m)$ and the number of codewords with weight w in a coset in the i-th $GL(m,2)$-*orbit*, respectively. Let $A_{sp,r,m}(w_L, w_R)$ denote the number of codewords of $RM(r,m)$ with left weight w_L and right weight w_R. Then, $A_{sp,r,m}(w_L, w_R)$ is given by

$$A_{sp,r,m}(w_L, w_R) = \sum_{i=1}^{s(r,m-1)} \nu_i(r,m-1) a_{r,m-1,i}(w_L) a_{r,m-1,i}(w_R). \qquad (3)$$

[1]A preliminary version of this paper was presented at IEICE technical meeting, Nagoya, Japan, Mar. 19, 1996.

The weight enumerator of $RM(r,m)$, $W_{r,m}(x,y) = \sum_{w=0}^{2^m} A_{r,m}(w)x^{2^m-w}y^w$ where $A_{r,m}(w)$ denotes the number of codewords with weight w in $RM(r,m)$, is given by

$$W_{r,m}(x,y) = \sum_{i=1}^{s(r,m)} \nu_i(r,m)W_{r,m,i}(x,y). \quad (4)$$

It follows from (2) and (4) that for $1 \le r \le m-1$,

$$W_{r,m}(x,y) = \sum_{i=1}^{s(r,m-1)} \nu_i(r,m-1)W^2_{r,m-1,i}(x,y). \quad (5)$$

If exactly one vector is chosen in a coset of each $GL(m,2)$-*orbit* in $RM(r,m)/RM(r-1,m)$, then such a vector is called a representative of the $GL(m,2)$-*orbit*.

(Fact 3) X.Hou has shown (I) $s(r,m)$ for $6 \le m \le 11$ and $1 \le r < m$, (II) $\nu_i(3,6)$ and $W_{3,6,i}(x,y)$ with $1 \le i \le 6 = s(3,6)$ and (III) a set of representatives of the $GL(m,2)$-*orbits* in $RM(3,m)/RM(2,m)$ with $6 \le m \le 8$ in [4], and $W_{3,8,i}(x,y)$ with $1 \le i \le 32 = s(3,8)$ in [5]. $W_{3,7,i}(x,y)$ with $1 \le i \le 12 = s(3,7)$ were presented in [11].

III. Computation of the Split Weight Distribution of RM(3,9)

In order to compute the split weight distributions of $RM(3,9)$ by using (3), we need to obtain $\nu_i(3,8)$ for $1 \le i \le 32 = s(3,8)$. Since $W_{3,8}(x,y)$ is known [6, 7, 8, 9], we have the following linear equations for $\nu_i(3,8)$ with $1 \le i \le 32$ from Eq.(4):

$$\sum_{i=1}^{32} \nu_i(3,8)a_{3,8,i}(w) = A_{3,8}(w), \quad \texttt{for} \quad 0 \le w \le 256. \quad (6)$$

The number of linearly independent equations in (6) turns out to be 16. Since formulas for $A_{3,9}(w)$ with $0 \le w \le 159$ and $A_{5,9}(w)$ with $0 \le w \le 39$ are known [9], we have the following additional equations from Eq.(5) and the MacWilliams identities [1]:

$$\sum_{i=1}^{32} \nu_i(3,8)\sum_{j=0}^{w} a_{3,8,i}(j)a_{3,8,i}(w-j) = A_{3,9}(w), \quad \texttt{for} \quad 0 \le w \le 159, \quad (7)$$

$$2^{-130}\sum_{h=0}^{512} P_w(h)\sum_{i=1}^{32} \nu_i(3,8)\sum_{j=0}^{h} a_{3,8,i}(j)a_{3,8,i}(h-j) = A_{5,9}(w), \quad \texttt{for} \quad 0 \le w \le 39, \quad (8)$$

where $P_w(h)$ is a Krawtchouk polynomial [1]. Since the number of linearly independent equations in (6) to (8) turned out to be 31, we used the following fact.

(Fact 4) Let P_m be the set of Boolean polynomials of variables $X_1, X_2, \cdots, X_m$. We use the representation of a codeword of a RM code by a Boolean polynomial [1]. Then $RM(r,m)$ is the set of polynomials of degree r or less in P_m. For $f \in P_m$, let $m(f)$ denote the minimum of n for which there are a transformation $T \in GL(m,2)$ and a polynomial $g \in P_n$ such that $T(f) \equiv g \pmod{RM(r-1,m)}$. For $1 \le i \le 12 = s(3,7)$, a representative f_i of the i-th $GL(m,2)$-*orbit* of $RM(3,8)/RM(2,8)$ shown in [Appendix in [4]] has $m(f_i)$ less than 8. For f and g in P_m such that $m' \triangleq \max\{m(f), m(g)\} < m$, $f + RM(r-1,m')$ and $g + RM(r-1,m')$ are not $GL(m',2)$-*equivalent* if and only if $f + RM(r-1,m)$ and $g + RM(r-1,m)$ are not $GL(m,2)$-*equivalent*. By using this property, we obtain the exact value of $m(f_i)$ with $1 \le i \le 12$ from the set of representatives shown in [4]. For $f \in RM(r,m)$, let $\nu(r,m,f)$ denote the number of cosets in the orbit containing $f + RM(r-1,m)$ in $RM(r,m)/RM(r-1,m)$. Then it is proved in [Appendix in [13]] that

$$\nu(r,m,f) = \nu(r,m(f),f)\prod_{i=0}^{m(f)} (2^{m-i}-1)/(2^{m(f)-i}-1). \quad (9)$$

For $1 \le i \le 12$, let the $GL(7,2)$-*orbit* in $RM(3,7)/RM(2,7)$ which contains $f_i + RM(2,7)$ be numbered i. $W_{3,7}(x,y)$ is listed in [10] and the weight distributions of twelve $GL(7,2)$-*orbits* in $RM(3,7)/RM(2,7)$ are tabulated in [11]. We used them for computing $\nu_i(3,7)$ by finding a correspondence of each representative shown in [11] to f_i in the same orbit. This correspondence has been easily verified by showing transformations in $GL(7,2)$. Then, we can compute $\nu_i(3,7)$ with $1 \le i \le 12$ by the following equations:

$$\sum_{i=1}^{12} \nu_i(3,7)a_{3,7,i}(w) = A_{3,7}(w), \quad \texttt{for} \quad 0 \le w \le 64, \quad (10)$$

and some equations of the following type:

$$\sum_{i=1}^{12} \nu_i(3,7)\sum_{j=0}^{w} a_{3,7,i}(j)a_{3,7,i}(w-j) = A_{3,8}(w). \quad (11)$$

By the value of $\nu_i(3,7)$ and the weight distributions of $RM(3,7)/RM(2,7)$, we computed the split weight distribution of RM(3,8) and tabulated in Table 1.

From $\nu_i(3,7)$ found above and the value of $m(f_i)$ with $1 \le i \le 12$, we computed $\nu_i(3,8)$ with $1 \le i \le 12$ by Eq.(9) and solved the remaining unknowns $\nu_i(3,8)$ with $13 \le i \le 32$ by using Eqs.(6), (7) and (8). The values of $\nu_i(3,8)$ with $1 \le i \le 32$ are shown in [Table

1 in [13]] together with the representatives f_i.
As a consequence, we computed the split weight distribution of RM(3,9) by the following equation and the computed result is tabulated in Table 2.

$$A_{sp,3,9}(w_L, w_R) = \sum_{i=1}^{s(3,8)} \nu_i(3,8) a_{3,8,i}(w_L) a_{3,8,i}(w_R). \quad (12)$$

By the relation to the split weight distribution, we can also compute the weight distribution of RM(3,9) naturally. Since $A_{r,m}(w) = A_{r,m}(2^m - w)$ for $0 \leq w \leq 2^{m-1}$, nonzero $A_{3,9}(w)$ with $0 \leq w \leq 2^8$ computed by Eq.(5) is tabulated in [12, 13, 14]. $A_{5,9}(w)$ with $0 \leq w \leq 2^8$ also have been computed by $A_{3,9}(w)$ and the MacWilliams identities. The weight distribution of $RM(5,9)$ is available in [12]. All the computational results were obtained by using *Mathematica*[2] on an *AlphaStation*[3] system at Nara Institute of Science and Technology.

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[2] *Mathematica* is a trademark of Wolfram Research, Inc.
[3] *AlphaStation* is a trademark of Digital Equipment Corp.

Tab. 1: The Split Weight Distribution of $RM(3,8)$.

	w_R						
	0	16	24	28	32	36	40
0	1	0	0	0	10668	0	0
16		755904	0	0	158739840	0	6772899840
24			2370514944	0	37928239104	0	3558142930944
28				214668673024	0	9131675090944	51322325827584
32					48998805855888	291288876318720	5970606507393024
36						14006255727149056	292172452576886784
40							8918091004218224640
44							
48							
52							
56							
60							
64							

	44	48	52
0	0	5291328	0
16	0	302880906048	1387089887232
24	9709629210624	215659967545344	1174865134485504
28	832253932339200	8214346312187904	51424970479239168
32	80551083931336704	855213311248318848	5026344914946883584
36	4960982465280737280	49612379672379654144	290466684336422780928
40	152384872467004391424	1513183410535617773568	8839378923357707698176
44	2623191062065000218624	25997312152409330417664	151872852220691028639744
48		257805711533910434451648	1506048369515032158928896
52			8798327668200779362074624
56			
60			
64			

	56	60	64
0	112881664	0	300503590
16	5858558361600	9709629210624	16197677210880
24	4231354951950336	8447377413242880	11605510170476544
28	173701105308401664	367388095256985600	476322704161505280
32	17441133232424337408	36752500102885539840	47359574918401134984
36	1012721565890204663808	2137114202215790149632	2740882570753804861440
40	30875218446629205368832	65138095034270346117120	83494735917873259413504
44	530534558162354989694976	1119288981123519794380800	1434605178013807420637184
48	5260870679020850002378752	11098999495642338828484608	14226132301123903129463424
52	30733366655725944415715328	64839457839179831917412352	83107828552812974774943744
56	107356126020070162865373184	226492478684070640021929984	290306275119112052939923456
60		477839844112846485995913216	612468405740944308295434240
64			785031333241834667972553124

Tab. 2: The Split Weight Distribution of $RM(3,9)$.

		$A_{sp}(w_R)$						
		0	32	48	56	64	68	72
	0	1	0	0	0	43180	0	0
	32		6217920	0	0	2698577280	0	0
	48			86354472960	0	2072507351040	0	0
$A_{sp}(w_L)$	56				23159447224320	14737830051840	0	781104992747520
	64					10380750418249360	0	74293401291325440
	68						181098455677009920	0
	72							1124055771836841984 0

		$A_{sp}(w_R)$			
		76	80	84	88
	0	0	0	0	0
	32	0	246727065600	0	0
	48	0	295418651996160	0	2475955448709120
	56	0	13087193086033920	0	235392621587988480
	64	0	173564615121666048 0	0	3555348226573860864 0
$A_{sp}(w_L)$	68	1267689189739069440	1810984556770099200	3929836488191115264 0	1521227027686883328 00
	72	4225630632463564800	2148694268736385843 20	4605937389385285632 00	6327216960173569474 560
	76	4314368875745299660 80	1457842568199929856 000	1469294027213906116 6080	1072591823438226653 18400
	80		3054521956499018496 0000	2294137126670793965 56800	2418723607993985752 104960
	84			3473523131984816614 932480	4039589215171154678 9068800
	88				5693103536354314426 51545600

		$A_{sp}(w_R)$		
		92	96	100
	0	0	89952576	0
	32	0	24837748810560	0
	48	0	51370548874444800	0
	56	25871207953858560 0	3440340095981322240	1267689189739069440 0
	64	7606135138434416640 0	7466405485978563936 00	3727006217832864153 600
	68	9618138981005996851 20	6262384597311003033 600	3875452621951309185 0240
	72	3153467408703773736 9600	2515565722078682284 03200	1551459724609907148 718080
$A_{sp}(w_L)$	76	8704854036073165514 34240	6591045696642754176 614400	4370261756940264803 4017280
	80	2174383635892594167 2509440	1792726482716116939 94065920	1194233064597573478 436044800
	84	4234412374867303234 69271040	3599491517308633715 048448000	2411303124529914061 9696865280
	88	6286638786778210930 859704320	5454091919869155247 5375206400	3655492617046560368 01061847040
	92	7161523178336529904 1741045760	6250754963786990387 87905781760	4191356262174860452 744774287360
	96		5467634976182061246 186051446976	3666402507748031378 3690849157120
	100			2458647986850734673 13454736998400

		$A_{sp}(w_R)$		
		104	108	112
	0	0	0	9594941440
	32	353707921244160	0	6221321649930240
	48	1488049224674181120	0	206754468037525094 40
	56	1033957226841322291 20	2970014673102962688 00	1378239539400167915 520
	64	2314252160129424162 8160	8731843138922710302 7200	2907929188386905426 32960
	68	2084472200595292500 78720	8584753162237286075 59680	2651551952204206681 620480
	72	8419703081348349768 499200	3379130246376562039 4557440	1049566931894695677 55223040
	76	2291249913833595645 35562240	9218561995594778256 44789760	2847358963307330261 074575360
$A_{sp}(w_R)$	80	6216935930022183256 058757120	2488235899896831782 1604331520	7687333196088869118 0467773440
	84	1248623665168174331 81647011840	4987010466431792927 97236674560	1540700281536499507 566854799360
	88	1889129013316500287 534231715840	7537825883958395334 742077603840	2328986054809215571 3856974356480
	92	2164309974333354740 1170638602240	8632751586401051406 8077522452480	2667376562585929727 9237827854336 0
	96	1892841187787335780 9200956178432 0	7549197108729347218 14003795886080	2332617055668054050 04710498623488 0
	100	1269288843183192480 8923135711641 60	5062288483321490042 40324227235840 0	1564186034151988669 31187524095180 80
	104	6552980346679791418 9839319695360 00	2613533002832197139 58123641870745 60	8075517242321956423 09173434331955 20
	108		1042368009883177751 67329727770787840	3220783684330045087 58194416632463360
	112			9951848491288448177 71204878033141760

		$A_{sp}(w_R)$	
		116	120
	0	0	28897705984
	32	0	24936408447713280
	48	0	8845598436058202112 0
	56	2209401159259521024 000	5928671401671194050 560
	64	6495639408222991810 56000	1276181936266353773 117440
	68	6296791311806622471 290880	1172032405935414633 8242560
	72	2496792278444374520 91555840	4663057623858040447 53960960
	76	6810502102440392812 092456960	1269950365848143053 7196011520
	80	1841764790812171038 41030307840	3436013168428705368 19640893440
$A_{sp}(w_L)$	84	3694583501689592327 832348917760	6893898609398823738 914289745920
	88	5586788561019120424 1219455549440	1042550117491327343 5867179909120 0
	92	6399387190333124972 7415378771968 0	1194214122880803164 6685200174284 80
	96	5596423404268320604 4822067688243 20	1044379174001404660 0846551803756544
	100	3752819154542477457 1501844630077440	7003321141565190366 2202137754992640
	104	1937476784318846414 3477709069090816 0	3615630908719354752 9107656660746240 0
	108	7727312755210175989 4472089722159104 0	1442030322747727065 6861348174181171 20
	112	2387641704919804413 4388405364169113 60	4455707280494506663 0314145886922342 40
	116	5728441121720099916 9134386260580761 60	1069011541805043022 3022480023061790720
	120		1994940541198539223 7289013558688874496

		$A_{sp}(w_R)$	
		124	128
	0	0	60273667110
	32	0	43750001797459200
	48	0	1495423882848249446 40
	56	5958139191773626368 000	1003927450013625483 2640
	64	1751692922381446152 192000	2139273580400385178 690440
	68	1711159954997648119 8243840	1943976743074890912 8908800
	72	6767706450566410601 81114880	7723219414580828606 30016000
	76	1848230880363649397 8309754880	2095313590826983542 3842304000
	80	4996179266798605619 40421017600	5662074038406975999 02021713920
	84	1002124656816873070 5362870599680	1135193940500750277 5661310771200
$A_{sp}(w_L)$	88	1515241012093799071 4233303597056 0	1716289518971354657 4697938812928 0
	92	1735582173141136514 7784784117760 00	1965771082103519881 8400117299609 60
	96	1517795504026501939 2987567331737600	1719090782120450786 8444118063452800
	100	1017794982313477610 9244864836665344 0	1152773344026629192 3511180823363584 0
	104	5254598287087459442 6771857219780608 0	5951483578389346405 8037789054205952 0
	108	2095712633049736786 3018197237104640 00	2373646643039941246 4062113929546956 80
	112	6475485836308528336 2668834977323417 60	7334293669766876309 3127952482669772 80
	116	1553601322894856151 5096819589395251200	1759640036147125965 8519876899574906880
	120	2899249051775539064 3976853823799951360	3283759522670168457 4905649985161789440
	124	4213497474191362956 3559760126961254400	4772291912370362206 9384880749490995200
	128		5405214087728376944 9335826035080929700