# New NCP-Functions and Their Properties*

by

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**Abstract.** Recently, Luo and Tseng proposed a class of merit functions for the nonlinear complementarity problem (NCP) and showed that it enjoys several interesting properties under some assumptions. In this paper, adopting a similar idea to Luo and Tseng's, we present new merit functions for the NCP, which can be decomposed into component functions. We show that these merit functions not only share many properties with the one proposed by Luo and Tseng but also enjoy additional favorable properties owing to their decomposable structure. In particular, we present fairly mild conditions under which these merit functions have bounded level sets.

**Key words:** Nonlinear complementarity problem, NCP-function, merit function, unconstrained optimization reformulation, error bound, bounded level sets.

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# 1. Introduction

The nonlinear complementarity problem (NCP) is to find a point $x \in \mathbb{R}^n$ such that

$$\langle x, F(x) \rangle = 0, \ x \in \mathbb{R}^n_+, \ F(x) \in \mathbb{R}^n_+,$$

where $F$ is a function from $\mathbb{R}^n$ into itself, $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$, and $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0, i = 1, \ldots, n\}$ [1, 2]. Recently, reformulations of the NCP as a minimization problem or a system of equations have drawn much attention [3]. A function which can constitute an equivalent minimization problem for the NCP is called a merit function. More precisely, a merit function is a function whose global minima on a set $X \subseteq \mathbb{R}^n$ are coincident with the solutions of the original NCP. To construct such a function, it is effective to take advantage of the Cartesian structure of the NCP. In particular, the class of functions defined below serves as a convenient tool for constructing a merit function [4, 5, 6, 7].

**Definition 1.1** *A function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is called an NCP-function if*

$$\phi(a, b) = 0 \Longleftrightarrow ab = 0, a \geq 0, b \geq 0.$$

By using an NCP-function $\phi$, we can construct a merit function $g : \mathbb{R}^n \to \mathbb{R}$ for the NCP as follows: If $\phi$ is nonnegative on $\mathbb{R}^2$, then define the function $g$ by

$$g(x) = \sum_{i=1}^{n} \phi(x_i, F_i(x)),$$

otherwise, by

$$g(x) = \sum_{i=1}^{n} \phi(x_i, F_i(x))^2.$$

It is easy to see that the NCP can be cast as the unconstrained optimization problem with the objective function $g$. There are many functions that belong to the class of NCP-functions. Among others, the following three NCP-functions have been well studied in the literature.

**Example 1.1**

**(i)** $\phi_{NR}(a, b) = \min\{a, b\}$.

**(ii)** $\phi_{MS}(a,b) = ab + \frac{1}{2\alpha}(\max\{0, a-\alpha b\}^2 - a^2 + \max\{0, b-\alpha a\}^2 - b^2), \quad \alpha > 1.$

**(iii)** $\phi_{FB}(a,b) = \sqrt{a^2+b^2} - a - b.$

The merit function based on the function $\phi_{NR}$ is called the natural residual. The function $\phi_{MS}$ is nonnegative on $\mathbb{R}^2$ and the merit function based on $\phi_{MS}$ is the implicit Lagrangian proposed by Mangasarian and Solodov [8]. The function $\phi_{FB}$ was first considered by Fischer [9] (and attributed to Burmeister). Recently, the merit function based on $\phi_{FB}$ has extensively been studied and has shown to have a number of favorable properties. Among the above three NCP-functions, the following useful relations hold.

**Proposition 1.1** *[10, 11] There exist positive constants $c_1, c_2, c_3, c_4$ such that*

$$c_1\phi_{NR}(a,b)^2 \le \phi_{FB}(a,b)^2 \le c_2\phi_{NR}(a,b)^2$$

*and*

$$c_3\phi_{NR}(a,b)^2 \le \phi_{MS}(a,b) \le c_4\phi_{NR}(a,b)^2$$

*for all $a, b \in \mathbb{R}$.*

This proposition indicates that if a merit function based on one of the three NCP-functions provides an error bound or has bounded level sets, then the same is true for the merit functions based on the other NCP-functions. To the authors' knowledge, for the merit functions based on these NCP-functions to provide global error bounds, not only the strong monotonicity (or the uniform $P$-property) but also the Lipschitz continuity of the function $F$ involved in the NCP are required [12, 13]. The question whether or not the merit functions based on these NCP-functions provide global error bounds without the Lipschitz continuity of $F$ remains unanswered.

Recently, Luo and Tseng [14] proposed a class of new merit functions $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \psi_0(\langle x, F(x)\rangle) + \sum_{i=1}^{n} \psi_i(-x_i, -F_i(x)), \tag{1}$$

where $\psi_0 : \mathbb{R} \to [0, \infty)$ and $\psi_i : \mathbb{R}^2 \to [0, \infty), i = 1, \ldots, n$, are continuous functions that vanish on the negative orthant only. The function $f$ is not based on an NCP-function in the sense of Definition 1.1. Nevertheless, the function $f$ enjoys several interesting properties under some assumptions as shown in [14]. Among others, $f$ turns out to provide a global error bound under the strong monotonicity of $F$ only, and it is convex on $\mathbb{R}^n$ under the assumption that $\langle x, F(x)\rangle$ and $-F_i(x), i = 1, \ldots, n$, are all convex functions of $x$.

We may naturally come up with the idea of constructing a new NCP-function by using a similar approach to Luo and Tseng's. In view of the structure of the function $f$, we notice that the second term is decomposable into $n$ component functions but the first term is not. So, for constructing a new merit function based on an NCP-function, we may substitute the decomposable function $\sum_{i=1}^n \psi_0(x_i F_i(x))$ for the function $\psi_0(\langle x, F(x)\rangle)$. In other words, we consider the following class of functions $g : \mathbb{R}^n \to \mathbb{R}$:

$$g(x) = \sum_{i=1}^{n} \phi_i(x_i, F_i(x)), \tag{2}$$

where $\phi_i : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\phi_i(a, b) = \psi_0(ab) + \psi_i(-a, -b) \tag{3}$$

for each $i = 1, \ldots, n$. It can be shown that $\phi_i$ is an NCP-function, and hence $g$ is a merit function for the NCP (see Lemma 2.1 and Theorem 2.1). Of course, the function $g$ is not the same as the merit function $f$. Nevertheless, we can expect that $g$ not only shares many properties with $f$ but also enjoys additional favorable properties owing to its decomposable structure. A possible cost to pay is that $g$ is in general nonconvex, while $f$ becomes convex when $F$ satisfies some conditions (see [14, Theorem 2.2]).

The following concepts [15, 16] will be useful throughout the paper.

**(a)** $F : \mathbb{R}^n \to \mathbb{R}^n$ is a $P_0$-function if

$$\max_{\substack{1 \le i \le n \\ x_i \ne y_i}} (x_i - y_i)(F_i(x) - F_i(y)) \ge 0 \quad \text{for all } x, y \in \mathbb{R}^n, x \ne y.$$

**(b)** $F : \mathbb{R}^n \to \mathbb{R}^n$ is a $P$-function if

$$\max_{1 \le i \le n} (x_i - y_i)(F_i(x) - F_i(y)) > 0 \quad \text{for all } x, y \in \mathbb{R}^n, x \neq y.$$

**(c)** $F : \mathbb{R}^n \to \mathbb{R}^n$ is a uniform $P$-function (with modulus $\mu$) if there is a $\mu > 0$ such that

$$\max_{1 \le i \le n} (x_i - y_i)(F_i(x) - F_i(y)) \ge \mu \|x - y\|^2 \quad \text{for all } x, y \in \mathbb{R}^n.$$

**(d)** An $n \times n$ matrix $M$ is a $P_0$-matrix if every principal minor of $M$ is nonnegative.

**(e)** An $n \times n$ matrix $M$ is a $P$-matrix if every principal minor of $M$ is positive.

It is known [16, Theorem 5.8] that if a $P_0$-function is differentiable, then its Jacobian matrices are $P_0$-matrices.

The paper is organized as follows. In the next section, we introduce a class of new merit functions. In Section 3 we show that the merit functions provide global error bounds for the NCP under the uniform $P$-property of the function $F$. In particular, this result does not rely on the Lipschitz continuity of $F$. In Section 4 we show that the merit functions have bounded level sets under fairly mild assumptions. In Section 5 we consider conditions under which any stationary point of the merit functions turns out to be a solution of the NCP. In Section 6 we present a simple descent method for solving the NCP. We conclude the paper with some final remarks in Section 7.

We shall use the following notation. For a vector $x \in \mathbb{R}^n$ and an index set $\alpha \subseteq \{1, 2, \ldots, n\}$, $x_\alpha$ represents the vector with elements $x_i, i \in \alpha$. By $\|\cdot\|$ and $\|\cdot\|_1$, we denote the Euclidean norm and $l_1$ norm, respectively. We denote the (transposed) Jacobian of $F$ at $x$ by $\nabla F(x)$, i.e., $\nabla F(x)$ is the matrix whose $(i, j)$th element is $\partial F_j(x)/\partial x_i$. For a vector $t \in \mathbb{R}^m$, $[t]_+$ denotes the vector whose $i$th element is $\max\{0, t_i\}$.

## 2. Unconstrained optimization reformulations

In this section, we show that the function $\phi_i$ defined by (3) is an NCP-function, so that the unconstrained optimization problem constituted from this function is equivalent to the NCP.

To begin with, following [14], we define a class of functions, to which the functions $\psi_i$ that constitute $\phi_i$ are supposed to belong.

**Definition 2.1** *The class of continuous functions $\psi : \mathbb{R}^m \to [0, \infty)$ such that*

$$\psi(t) = 0 \Longleftrightarrow t \leq 0$$

*is denoted by $\Psi^m$.*

The following lemma shows the most fundamental property of the function $\phi_i$ defined by (3).

**Lemma 2.1** *For any $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2$, the function $\phi_i$ defined by (3) is a nonnegative NCP-function.*

**Proof.** Since $\psi_0$ and $\psi_i$ are nonnegative by definition, $\phi_i$ is nonnegative. Now, we prove that $\phi_i$ is an NCP-function. First, suppose that $(a, b)$ is a vector such that $a \geq 0, b \geq 0$ and $ab = 0$. Then, by the definition of $\Psi^m$, we have $\psi_0(ab) = 0$ and $\psi_i(-a, -b) = 0$, which yields $\phi_i(a, b) = 0$. Next, suppose that $\phi_i(a, b) = 0$. Then, by the nonnegativity of $\psi_0$ and $\psi_i$, we must have $\psi_0(ab) = 0$ and $\psi_i(-a, -b) = 0$. Moreover, by the definition of $\Psi^m$, we obtain $ab \leq 0$, $a \geq 0$ and $b \geq 0$, from which it follows that $ab = 0$. Hence, $\phi_i$ is an NCP-function. □

The following theorem establishes the equivalence between the NCP and the unconstrained minimization problem

$$\min_{x \in \mathbb{R}^n} \quad g(x),$$

where $g$ is defined by (2).

**Theorem 2.1** *Let $\phi_i : \mathbb{R}^2 \to \mathbb{R}$ be defined by (3) with arbitrary $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2$ for $i = 1, \ldots, n$, and let $g : \mathbb{R}^n \to \mathbb{R}$ be defined by (2). Then $g$ is nonnegative on $\mathbb{R}^n$. Moreover, $g(x) = 0$ if and only if $x$ is a solution of the NCP.*

**Proof.** By Lemma 2.1, $\phi_i$ are nonnegative NCP-functions for all $i = 1, \ldots, n$. Moreover, $g(x) = 0$ if and only if $\phi_i(x_i, F_i(x)) = 0$ for all $i = 1, \ldots, n$. Hence the theorem follows. □

We provide examples of functions that belong to $\Psi^m, m = 1, 2$.

**Definition 2.2** *For $p > 0$, let $\psi_0^p : \mathbb{R} \to [0, \infty)$ be defined by*

$$\psi_0^p(t) = \max\{0, t\}^p,$$

*and let $\psi_I^p, \psi_{II}^p, \psi_{FB}^p, \psi_M^p : \mathbb{R}^2 \to [0, \infty)$ be defined, respectively, by*

$$\begin{aligned}
\psi_I^p(t) &= \|[t]_+\|_1^p \\
&= (\max\{0, t_1\} + \max\{0, t_2\})^p, \\
\psi_{II}^p(t) &= \|[t]_+\|^p \\
&= \left(\max\{0, t_1\}^2 + \max\{0, t_2\}^2\right)^{p/2}, \\
\psi_{FB}^p(t) &= \left([\phi_{FB}(-t)]_+\right)^p \\
&= \max\left\{0, \sqrt{t_1^2 + t_2^2} + t_1 + t_2\right\}^p, \\
\psi_M^p(t) &= \max\{0, t_1, t_2\}^p.
\end{aligned}$$

It is easy to observe that $\psi_0^p \in \Psi^1$ and $\psi_I^p, \psi_{II}^p, \psi_{FB}^p, \psi_M^p \in \Psi^2$. Moreover, for $p > 1$, the functions $\psi_0^p$, $\psi_{II}^p$ and $\psi_{FB}^p$ are continuously differentiable up to $(p-1)$th order.

Using the functions given in Definition 2.2, we present some examples of the NCP-function $\phi_i$ defined by (3).

**Definition 2.3** *For $p > 0$, let $\phi_I^p, \phi_{II}^p, \phi_{FB}^p, \phi_M^p : \mathbb{R}^2 \to [0, \infty)$ be defined, respectively, by*

$$\begin{aligned}
\phi_I^p(a, b) &= \psi_0^p(ab) + \psi_I^p(-a, -b), \\
\phi_{II}^p(a, b) &= \psi_0^p(ab) + \psi_{II}^p(-a, -b), \\
\phi_{FB}^p(a, b) &= \psi_0^p(ab) + \psi_{FB}^p(-a, -b), \\
\phi_M^p(a, b) &= \psi_0^p(ab) + \psi_M^p(-a, -b).
\end{aligned}$$

For $p > 1$, the functions $\phi_{II}^p$ and $\phi_{FB}^p$ are continuously differentiable up to $(p-1)$th order.

Now we display four special cases of $g$, which correspond to the functions given in Definition 2.3.

**Definition 2.4** *For $p > 0$, let $g_I^p, g_{II}^p, g_{FB}^p, g_M^p : \mathbb{R}^n \to [0, \infty)$ be defined, respectively, by*

$$\begin{aligned} g_I^p(x) &= \sum_{i=1}^n \phi_I^p(x_i, F_i(x)), \\ g_{II}^p(x) &= \sum_{i=1}^n \phi_{II}^p(x_i, F_i(x)), \\ g_{FB}^p(x) &= \sum_{i=1}^n \phi_{FB}^p(x_i, F_i(x)), \\ g_M^p(x) &= \sum_{i=1}^n \phi_M^p(x_i, F_i(x)). \end{aligned}$$

Note that for $p > 1$, the functions $g_{II}^p$ and $g_{FB}^p$ are continuously differentiable up to $(p-1)$th order, provided that $F$ is continuously differentiable up to $(p-1)$th order.

# 3. Global error bounds

In this section, we show that if $F$ is a uniform $P$-function and if $\psi_0 \in \Psi^1$ and $\psi_1, \ldots, \psi_n \in \Psi^2$ satisfy suitable assumptions, then the merit function $g$ defined by (2) provides a global error bound for the NCP. To the authors' knowledge, this is the first time that a global error bound result is established for a merit function under the uniform $P$-function property only. For example, for the squared Fischer-Burmeister function and the implicit Lagrangian, a similar result is currently known only when $F$ is not only a uniform $P$-function but also a globally Lipschitz-continuous function [4, 12, 17].

The error bound results established below rely on the following growth assumption on $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2, i = 1, \ldots, n$.

**Assumption 3.1** *The functions $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2, i = 1, \ldots, n$, have growth of order $p > 0$ in the sense that there exists a positive constant $\tau$ such that, for each $i = 0, 1, \ldots, n$,*

*$\psi_i$ satisfies*

$$\psi_i(t) \geq \tau \|[t]_+\|^p \quad \textit{for all} \quad t.$$

The next proposition shows that all the functions given in Definition 2.2 have growth of order $p$.

**Proposition 3.1** *The functions $\psi_0^p$, $\psi_I^p$, $\psi_{II}^p$, $\psi_{FB}^p$ and $\psi_M^p$ have growth of order $p$.*

**Proof.** It is easy to show that the functions $\psi_0^p$, $\psi_I^p$, $\psi_{II}^p$ and $\psi_M^p$ have growth of order $p$. Here we only prove that $\psi_{FB}^1$ has growth of order 1. Then it will be easy to verify that $\psi_{FB}^p$ has growth of order $p$ for any $p > 0$. If $t_1$ and $t_2$ are both nonnegative, then we have

$$\begin{aligned} \psi_{FB}^1(t) &= \sqrt{t_1^2 + t_2^2} + t_1 + t_2 \\ &\geq \sqrt{t_1^2 + t_2^2} \\ &= \|[t]_+\|. \end{aligned}$$

If $t_1$ and $t_2$ are both nonpositive, then we have $\psi_{FB}^1(t) = 0 = \|[t]_+\|$. Finally, without loss of generality, we suppose $t_1 > 0$ and $t_2 < 0$. Then we have

$$\begin{aligned} \psi_{FB}^1(t) &= \sqrt{t_1^2 + t_2^2} + t_1 + t_2 \\ &\geq t_1 \\ &= \|[t]_+\|. \end{aligned}$$

The proof is complete. □

The next result relates the function $g$ to the function $g_I^1$. Recall that the function $g_I^1 : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\begin{aligned} g_I^1(x) &= \sum_{i=1}^n \phi_I^1(x_i, F_i(x)) \\ &= \sum_{i=1}^n \left( [x_i F_i(x)]_+ + [-F_i(x)]_+ + [-x_i]_+ \right). \end{aligned}$$

**Lemma 3.1** *Let Assumption 3.1 be satisfied and the function $g$ be defined by (2). Then, there exists a positive constant $\kappa$ such that*

$$g_I^1(x) \leq \kappa g(x)^{1/p} \quad \textit{for all } x \in \mathbb{R}^n. \tag{4}$$

**Proof.** The inequality can be shown to hold in a way similar to [14, Lemma 3.1]. □

The inequality (4) shows that if $g_I^1$ provides a global error bound, then $g$ also provides a global error bound.

We are now in a position to prove the main theorem of this section.

**Theorem 3.1** *Suppose that $F$ is a uniform $P$-function. Let Assumptions 3.1 be satisfied and the function $g$ be defined by (2). Then there exists a positive constant $\tau$ such that*

$$\|x - x^*\|^2 \leq \tau g(x)^{1/p} \quad \textit{for all } x \in \mathbb{R}^n, \tag{5}$$

*where $x^*$ is the unique solution of the NCP.*

**Proof.** Since $F$ is a uniform $P$-function, there exists a constant $\mu > 0$ such that

$$\mu\|x - x^*\|^2 \leq \max_{1\leq i\leq n}(x_i - x_i^*)(F_i(x) - F_i(x^*)) \tag{6}$$

for all $x \in \mathbb{R}^n$. Also, since $x^*$ solves the NCP, we have $x_i^* \geq 0, F_i(x^*) \geq 0$ and $x_i^* F_i(x^*) = 0$ for all $i$. Therefore we have, for each $i$ and all $x \in \mathbb{R}^n$,

$$\begin{aligned}(x_i - x_i^*)(F_i(x) - F_i(x^*)) &= x_i F_i(x) - x_i^* F_i(x) - F_i(x^*)x_i \\ &\leq [x_i F_i(x)]_+ + x_i^*[-F_i(x)]_+ + F_i(x^*)[-x_i]_+ \\ &\leq \tau_i\left([x_i F_i(x)]_+ + [-F_i(x)]_+ + [-x_i]_+\right) \\ &= \tau_i \phi_I^1(x_i, F_i(x)),\end{aligned} \tag{7}$$

where $\tau_i := \max\{1, x_i^*, F_i(x^*)\} > 0$. Let $\hat{\tau} := \max_{1\leq i\leq n}\tau_i$. Then, combining (6) and (7), we obtain

$$\mu\|x - x^*\|^2 \leq \max_{1\leq i\leq n}\tau_i\phi_I^1(x_i, F_i(x)) \leq \hat{\tau}\sum_{i=1}^n \phi_I^1(x_i, F_i(x)) = \hat{\tau} g_I^1(x).$$

The desired inequality (5) then follows from Lemma 3.1. □

Proposition 3.1 and Theorem 3.1 indicate that the functions $g_I^p$, $g_{II}^p$, $g_{FB}^p$ and $g_M^p$ all satisfy the inequality (5) and hence provide global error bounds if $F$ is a uniform $P$-function.

# 4. Bounded level sets

In this section we consider conditions which guarantee that the level sets

$$\mathcal{L}(c) := \{x \in \mathbb{R}^n \,|\, g(x) \leq c\}$$

are bounded for all $c \geq 0$. To establish the bounded level set results, we need the following assumption on the functions $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2, i = 1, \ldots, n$, which constitute the merit function $g$.

**Assumption 4.1** *The functions $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2, i = 1, \ldots, n$, have the property that, for each $i = 0, 1, \ldots, n$, $\|[t]_+\| \to \infty$ implies $\psi_i(t) \to \infty$.*

Note that every merit function $g$ given in Definition 2.4 is constituted from functions $\psi_i$ satisfying Assumption 4.1, cf. Proposition 3.1.

We first deal with monotone problems. Before stating the result, we note that the particular level set $\mathcal{L}(0)$ is equal to the solution set of the NCP. Hence, if the NCP has an unbounded solution set, the level sets $\mathcal{L}(c)$ cannot be bounded for any $c \geq 0$ since $\mathcal{L}(0) \subseteq \mathcal{L}(c)$ for all $c \geq 0$. Since it can happen that the monotone complementarity problem has an unbounded solution set (e.g., take $F(x) \equiv 0$), we need an additional assumption, namely that the NCP is strictly feasible, i.e., that there exists a vector $\hat{x} > 0$ with $F(\hat{x}) > 0$.

**Theorem 4.1** *Assume that $F$ is monotone and the NCP is strictly feasible. Let $g$ be defined by (2) and Assumption 4.1 be satisfied. Then the level sets $\mathcal{L}(c)$ are bounded for all $c \geq 0$.*

**Proof.** Assume there exists an unbounded sequence $\{x^k\} \subseteq \mathcal{L}(c)$ for some $c \geq 0$. We first note that there is no index $j$ such that $x_j^k \to -\infty$ or $F_j(x^k) \to -\infty$ on a subsequence, because otherwise we would have $\psi_j(-x_j^k, -F_j(x^k)) \to \infty$ by Assumption 4.1, which in turn would imply $g(x^k) \to \infty$ on a subsequence. Hence all components of the two sequences $\{x^k\}$ and $\{F(x^k)\}$ are bounded from below. On the other hand, the sequence $\{x^k\}$ is unbounded

by assumption. Hence there exists an index $j$ such that $x_j^k \to \infty$ on a subsequence. Now let $\hat{x}$ satisfy the strict feasibility condition for the NCP. Since $F$ is monotone, we have

$$\langle x^k, F(\hat{x})\rangle + \langle \hat{x}, F(x^k)\rangle \leq \langle x^k, F(x^k)\rangle + \langle \hat{x}, F(\hat{x})\rangle. \tag{8}$$

Since $\hat{x} > 0$ and $F(\hat{x}) > 0$, it follows immediately from the previous remarks that the left-hand side of (8) tends to infinity. Hence $\langle x^k, F(x^k)\rangle \to \infty$ on a subsequence. Thus, there is an index $l$ such that $x_l^k F_l(x^k) \to \infty$ on a subsequence. By Assumption 4.1, this implies that $\psi_0(x_l^k F_l(x^k))$ and therefore $g(x^k)$ itself tend to infinity on a subsequence. This is the desired contradiction to $\{x^k\} \subseteq \mathcal{L}(c)$. □

Note that this bounded level set result for the monotone NCP is also true for the merit function proposed by Luo and Tseng [14]. This follows immediately from the definition of the function $f$ in (1) and the proof given for Theorem 4.1. However, for the merit functions constituted by the NCP-functions of Example 1.1, this result does not hold. In fact, consider the simple example with $n = 1$, where $F(x) \equiv 1$. Then $F$ is monotone and the NCP is strictly feasible. However, for the merit function constituted by the NCP-function $\phi_{NR}$, we have

$$\mathcal{L}_{NR}(1) := \{x \in \mathbb{R} \mid \min\{x, F(x)\}^2 \leq 1\} = [-1, \infty),$$

which is unbounded. By Proposition 1.1, this also implies that the implicit Lagrangian and the squared Fischer-Burmeister function, which are constituted from $\phi_{MS}$ and $\phi_{FB}$ respectively, have unbounded level sets for this example.

We next present another condition which guarantees the boundedness of the level sets. This result is a straightforward modification of the corresponding result by Luo and Tseng [14, Theorem 4.1].

**Theorem 4.2** *Let $g$ be defined by (2) and Assumption 4.1 be satisfied. Suppose that, for any sequence $\{x^k\}$ such that $\|x^k\| \to \infty$, $[-x^k]_+/\|x^k\| \to 0$ and $[-F(x^k)]_+/\|x^k\| \to 0$, we have*

$$\liminf_{k\to\infty} \frac{\max_i\{x_i^k F_i(x^k)\}}{\|x^k\|} > 0.$$

*Then the level sets $\mathcal{L}(c)$ are bounded for all $c \geq 0$.*

**Proof.** The theorem can be proved in a way similar to [14, Theorem 4.1]. □

We can easy verify that

$$\liminf_{k\to\infty} \frac{\langle x^k, F(x^k)\rangle}{\|x^k\|} > 0 \implies \liminf_{k\to\infty} \frac{\max_i\{x_i^k F_i(x^k)\}}{\|x^k\|} > 0.$$

So, the assumptions on $F$ in Theorem 4.2 are weaker than those in [14, Theorem 4.1], which hold whenever $F$ is a uniform $P$-function or an $R_0$-function in the sense of Chen and Harker [18]. In particular, the level sets are bounded if $F$ is an affine mapping such that $F(x) = Mx + q$ with an $R_0$-matrix $M$.

# 5. Stationary points

Luo and Tseng [14] gave conditions under which the merit function $f$ defined by (1) is convex. However, the merit function $g$ given by (2) is not necessarily convex under such conditions. Nevertheless, since commonly used minimization methods can only find a stationary point or a local minimum, it is important to investigate conditions under which any stationary point of $g$ is a solution of the NCP. In this section, we focus our attention to such conditions.

In what follows, we suppose the functions $\phi_i$ consist of continuously differentiable functions $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2$. Then $\phi_i$ are also continuously differentiable and

$$\frac{\partial \phi_i(a,b)}{\partial a} = b\psi_0'(ab) - \frac{\partial \psi_i(-a,-b)}{\partial a}, \tag{9}$$

$$\frac{\partial \phi_i(a,b)}{\partial b} = a\psi_0'(ab) - \frac{\partial \psi_i(-a,-b)}{\partial b}. \tag{10}$$

Moreover, if $F$ is continuously differentiable, then the function $g$ defined by (2) is also continuously differentiable.

We define the following classes of continuously differentiable functions, which are given by Luo and Tseng [14].

**Definition 5.1** *$\Psi_+^m$ denotes the class of continuously differentiable functions $\psi \in \Psi^m$ such that*

**C.1** *$\langle t, \nabla\psi(t)\rangle \geq 0$ for all $t \in \mathbb{R}^m$;*

**C.2** *$\nabla\psi(t) \geq 0$ for all $t \in \mathbb{R}^m$;*

**C.3** *$\nabla\psi(t) = 0 \Longleftrightarrow t \leq 0$.*

*Moreover, $\Psi_{++}^m$ denotes the class of functions $\psi \in \Psi_+^m$ such that*

**C.4** *$\nabla\psi(t) > 0$ for all $t \not\leq 0$.*

The functions $\psi_0^p$, $\psi_{II}^p$ and $\psi_{FB}^p$ with $p > 1$ belong to $\Psi_+^1$, $\Psi_+^2$ and $\Psi_+^2$, respectively. However, the function $\psi_{II}^p$ with $p > 1$ does not belong to $\Psi_{++}^2$, whereas the function $\psi_{FB}^p$ with $p > 1$ belongs to $\Psi_{++}^2$. Note that $\psi_I^p$ and $\psi_M^p$ are not differentiable and hence belong to neither $\Psi_+^2$ nor $\Psi_{++}^2$.

The next lemma shows some important properties of the NCP-function $\phi_i$ constituted by $\psi_i$ belonging to the classes of functions defined above.

**Lemma 5.1** *Let the NCP-function $\phi_i$ be defined by (3) with $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_+^2$. Then, $\phi_i$ has the following properties.*

**(P.1)** *$\phi_i(a,b) = 0 \Longleftrightarrow \nabla\phi_i(a,b) = 0$;*

**(P.2)** $\dfrac{\partial\phi_i(a,b)}{\partial a}\dfrac{\partial\phi_i(a,b)}{\partial b} \geq 0$ *for all $(a,b)^T \in \mathbb{R}^2$.*

*Moreover, if $\psi_i \in \Psi_{++}^2$, then we have*

**(P.3)** $\dfrac{\partial\phi_i(a,b)}{\partial a} = 0 \Longleftrightarrow \dfrac{\partial\phi_i(a,b)}{\partial b} = 0.$

**Proof. (P.1)** First, suppose that $\phi_i(a,b) = 0$. Then, since $\psi_i$ are nonnegative functions, we have

$$ab \leq 0, \quad \begin{pmatrix} -a \\ -b \end{pmatrix} \leq 0.$$

Hence, C.3 yields $\psi_0'(ab) = 0$ and $\nabla\psi_i(-a,-b) = 0$. It then follows from (9) and (10) that $\nabla\phi_i(a,b) = 0$. Next, suppose that $\nabla\phi_i(a,b) = 0$. Then, by (9) and (10), we obtain

$$b\psi_0'(ab) = \frac{\partial\psi_i(-a,-b)}{\partial a} \tag{11}$$

and

$$a\psi_0'(ab) = \frac{\partial\psi_i(-a,-b)}{\partial b}. \tag{12}$$

Let the both sides of (11) and (12) be nonzero. Then, since $\psi_0'(ab) \geq 0$ and $\nabla\psi_i(-a,-b) \geq 0$ by C.2, we must have $a > 0$ and $b > 0$ from (11) and (12). It then follows from C.3 that $\nabla\psi_i(-a,-b) = 0$, which is a contradiction. Therefore, the both sides of at least one of (11) and (12) are zero. Without loss of generality, we assume the both sides of (11) are zero. Then, by C.3, we obtain $ab \leq 0$ or $b = 0$. It can easily be verified by considering the two cases $ab \leq 0$ and $b = 0$ separately that the both sides of (12) are zero. Hence, we have $\nabla\psi_i(-a,-b) = 0$ and $\psi_0'(ab) = 0$. It then follows from C.3 that $a \geq 0$, $b \geq 0$ and $ab \leq 0$, which yield $a \geq 0$, $b \geq 0$ and $ab = 0$. Since $\phi_i$ is an NCP-function, we obtain $\phi_i(a,b) = 0$.

**(P.2)** By C.1 and C.2, we have

$$ab\psi_0'(ab)^2 = (ab\psi_0'(ab))\,\psi_0'(ab) \geq 0. \tag{13}$$

We also have from C.2

$$\frac{\partial\psi_i(-a,-b)}{\partial a}\frac{\partial\psi_i(-a,-b)}{\partial b} \geq 0. \tag{14}$$

Furthermore, C.1 and C.2 yield

$$\begin{aligned}\psi_0'(ab)&\left(-a\frac{\partial\psi_i(-a,-b)}{\partial a} - b\frac{\partial\psi_i(-a,-b)}{\partial b}\right)\\ &= \psi_0'(ab)\left\langle(-a,-b)^T, \nabla\psi_i(-a,-b)\right\rangle \geq 0.\end{aligned} \tag{15}$$

From (9) and (10), we have

$$\begin{aligned}\frac{\partial\phi_i(a,b)}{\partial a}\frac{\partial\phi_i(a,b)}{\partial b} &= ab\psi_0'(ab)^2 + \frac{\partial\psi_i(-a,-b)}{\partial a}\frac{\partial\psi_i(-a,-b)}{\partial b}\\ &\quad +\psi_0'(ab)\left(-a\frac{\partial\psi_i(-a,-b)}{\partial a} - b\frac{\partial\psi_i(-a,-b)}{\partial b}\right),\end{aligned} \tag{16}$$

which shows (P.2) by (13)-(15).

**(P.3)** Suppose that $\frac{\partial \phi_i(a,b)}{\partial a} = 0$. Then, by (16), we have

$$\begin{aligned} 0 &= ab\psi_0'(ab)^2 + \frac{\partial \psi_i(-a,-b)}{\partial a}\frac{\partial \psi_i(-a,-b)}{\partial b} \\ &\quad + \psi_0'(ab)\left(-a\frac{\partial \psi_i(-a,-b)}{\partial a} - b\frac{\partial \psi_i(-a,-b)}{\partial b}\right). \end{aligned}$$

Hence, by (13)-(15), every term of the right-hand side must be zero. Moreover, C.3 and the first term being zero yield $ab \leq 0$. If either $a$ or $b$ is negative, then the second term is positive by C.4. This is a contradiction. So, we must have $a \geq 0$ and $b \geq 0$. Hence, we obtain $ab = 0$, $a \geq 0$ and $b \geq 0$, which yield $\phi_i(a,b) = 0$. Then, by (P.1) proved earlier, we have $\frac{\partial \phi_i(a,b)}{\partial b} = 0$. We can prove the converse in a similar way. □

Note that the functions $\phi_{II}^p$ and $\phi_{FB}^p$ with $p > 1$ enjoy properties (P.1) and (P.2). Moreover, the function $\phi_{FB}^p$ with $p > 1$ has property (P.3), whereas the function $\phi_{II}^p$ with $p > 1$ does not.

Now, we give a necessary and sufficient condition for a stationary point of $g$ to be a solution of the NCP. To this end, we define the index sets

$$\begin{aligned} \mathcal{P}(x) &:= \left\{ i \;\middle|\; \frac{\partial \phi_i(x_i, F_i(x))}{\partial b} > 0 \right\}, \\ \mathcal{C}(x) &:= \left\{ i \;\middle|\; \frac{\partial \phi_i(x_i, F_i(x))}{\partial b} = 0 \right\}, \\ \mathcal{N}(x) &:= \left\{ i \;\middle|\; \frac{\partial \phi_i(x_i, F_i(x))}{\partial b} < 0 \right\}, \\ \mathcal{R}(x) &:= \mathcal{P}(x) \cup \mathcal{N}(x). \end{aligned}$$

For simplicity, we shall denote these sets by $\mathcal{P}$, $\mathcal{C}$, $\mathcal{N}$ and $\mathcal{R}$. The point $x$ under consideration will always be clear from the context. Using these index sets, we define regularity conditions. Similar concepts of regularity have been considered by De Luca et al. [19], Facchinei and Kanzow [20] and Luo and Tseng [14] for different merit functions.

**Definition 5.2** *A point $x \in \mathbb{R}^n$ is said to be regular if for every nonzero vector $z \in \mathbb{R}^n$ such that*

$$z_{\mathcal{C}} = 0, \quad z_{\mathcal{P}} > 0, \quad z_{\mathcal{N}} < 0,$$

*there exists a vector* $y \in \mathbb{R}^n$ *such that*

$$y_{\mathcal{P}} \geq 0, \quad y_{\mathcal{N}} \leq 0, \quad y_{\mathcal{R}} \neq 0$$

*and*

$$\langle y, \nabla F(x)z \rangle \geq 0.$$

*Moreover, a point* $x \in \mathbb{R}^n$ *is said to be strictly regular if for every nonzero vector* $z \in \mathbb{R}^n$ *such that*

$$z_{\mathcal{C}} = 0, \quad z_{\mathcal{P}} > 0, \quad z_{\mathcal{N}} < 0,$$

*there exists a vector* $y \in \mathbb{R}^n$ *such that*

$$y_{\mathcal{C}} = 0, \quad y_{\mathcal{P}} \geq 0, \quad y_{\mathcal{N}} \leq 0$$

*and*

$$\langle y, \nabla F(x)z \rangle > 0.$$

The first regularity in Definition 5.2 is similar to that of [19], whereas the second strict regularity is similar to that of [20], where it is simply called regular. We next present a sufficient condition for a vector $x \in \mathbb{R}^n$ to be (strictly) regular.

**Lemma 5.2** *Let* $x \in \mathbb{R}^n$. *If* $\nabla F(x)$ *is a* $P_0$*-matrix (respectively,* $P$*-matrix), then* $x$ *is regular (respectively, strictly regular).*

**Proof.** We only consider the case where $\nabla F(x)$ is a $P_0$-matrix. The proof is similar for the case where $\nabla F(x)$ is a $P$-matrix, see also [20].

So let $z \in \mathbb{R}^n$ be a nonzero vector with $z_{\mathcal{C}} = 0, z_{\mathcal{P}} > 0$ and $z_{\mathcal{N}} < 0$. Since $\nabla F(x)$ is a $P_0$-matrix, there exists an index $j \in \mathcal{R}$ such that $z_j[\nabla F(x)z]_j \geq 0$. Now choose $y \in \mathbb{R}^n$ in such a way that $y_j = z_j$ and $y_i = 0$ for all $i \neq j$. Then

$$\langle y, \nabla F(x)z \rangle = \sum_{i=1}^{n} y_i[\nabla F(x)z]_i = y_j[\nabla F(x)z]_j = z_j[\nabla F(x)z]_j \geq 0,$$

i.e., $x$ is regular. □

Similarly to Lemma 5.2, one can show that if $\nabla F(x)$ is (strictly) semimonotone and $\mathcal{N} = \emptyset$, then $x$ is (strictly) regular, see [15] for the definition of a (strictly) semimonotone matrix.

The following theorem shows that the (strict) regularity provides a necessary and sufficient condition for a stationary point of $g$ to be a solution of the NCP.

**Theorem 5.1** *Let the NCP-functions $\phi_i$ be defined by (3) with $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_{++}^2$ (respectively, $\psi_i \in \Psi_+^2$) for all $i = 1, \ldots, n$, and let $g$ be defined by (2). Suppose that $x$ is a stationary point of $g$. Then, $x$ is a solution of the NCP if and only if $x$ is regular (respectively, strictly regular).*

**Proof.** By using Lemma 5.1, it can be proved in a way similar to the proof of [19, Theorem 4.2] for the case where $\phi_i$ are constituted by $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_{++}^2$, and the proof of [20, Theorem 3.3] for the case where $\phi_i$ are constituted by $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_+^2$. □

Theorem 5.1 indicates that a necessary and sufficient condition for any stationary point $x$ of $g_{FB}^p$ with $p > 1$ to be a solution of the NCP is that $x$ is a regular point, whereas such a condition for $g_{II}^p$ with $p > 1$ is that $x$ is a strictly regular point.

From Lemma 5.2, we obtain the following corollary, which gives sufficient conditions under which any stationary point of $g$ solves the NCP.

**Corollary 5.1** *Let the NCP-functions $\phi_i$ be defined by (3) with $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_{++}^2$ (respectively, $\psi_i \in \Psi_+^2$) for all $i = 1, \ldots, n$, and let $g$ be defined by (2). If $F$ is a $P_0$-function (respectively, $\nabla F(x)$ is a $P$-matrix for any $x$), then any stationary point of $g$ is a solution of the NCP.* □

Corollary 5.1 indicates that a sufficient condition for any stationary point of $g_{FB}^p$ with $p > 1$ to be a solution of the NCP is that $F$ is a $P_0$-function, whereas such a condition for $g_{II}^p$ with $p > 1$ is that $\nabla F(x)$ is a $P$-matrix.

As an application of the previous results, we obtain the following corollary.

**Corollary 5.2** *Assume that $F$ is monotone and the NCP is strictly feasible. Then the NCP has a nonempty and bounded solution set.*

**Proof.** We consider $g^2_{FB}$. Note that, by Theorem 4.1, the level set $\mathcal{L}(c)$ of $g^2_{FB}$ is nonempty and compact whenever $c \geq g^2_{FB}(x)$ for some $x \in \mathbb{R}^n$. Hence, by the Bolzano-Weierstrass Theorem, the function $g^2_{FB}$ has a global minimum $x^*$. In particular, $x^*$ is a stationary point of $g^2_{BF}$. Since $F$ is monotone, $\nabla F(x^*)$ is positive semidefinite and therefore a $P_0$-matrix. Hence $g^2_{BF}(x^*) = 0$ by Corollary 5.1, i.e., $x^*$ solves the NCP. The boundedness of the solution set follows from the fact that the level set $\mathcal{L}(0)$ is bounded. □

We note that the results stated in Corollary 5.2 are known to hold under even weaker conditions [1, Theorem 3.4]. But the above proof is completely different from classical proofs and, in particular, quite elementary.

# 6. Descent methods

Similar to, e.g., [5, 21, 14, 17], we present in this section a simple descent method which makes no use of the derivatives of $F$. This method is therefore applicable in particular to large-scale problems.

Throughout this section, we will use the following notation:

$$\frac{\partial \phi(x, F(x))}{\partial a} := \left( \frac{\partial \phi_1(x_1, F_1(x))}{\partial a}, \ldots, \frac{\partial \phi_n(x_n, F_n(x))}{\partial a} \right)^T \in \mathbb{R}^n$$

and

$$\frac{\partial \phi(x, F(x))}{\partial b} := \left( \frac{\partial \phi_1(x_1, F_1(x))}{\partial b}, \ldots, \frac{\partial \phi_n(x_n, F_n(x))}{\partial b} \right)^T \in \mathbb{R}^n.$$

As a search direction, we will take the vector

$$d(x) := -\frac{\partial \phi(x, F(x))}{\partial b}. \tag{17}$$

The following lemma shows that this direction is a descent direction of the merit function $g$ with $\psi_0 \in \Psi^1_+$ and $\psi_i \in \Psi^2_{++}, i = 1, \ldots, n$, if $\nabla F(x)$ is positive semidefinite.

**Lemma 6.1** *Let the NCP-functions $\phi_i$ be defined by (3) with $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_{++}^2$ for all $i = 1, \ldots, n$, and let $g$ be defined by (2). If $\nabla F(x)$ is positive semidefinite, then the direction $d(x)$ given by (17) is a descent direction for $\Psi$ at $x$ as long as $x$ is not a solution of the NCP.*

**Proof.** Suppose that $x$ is not a solution of the NCP. We recall that the gradient of $g$ can be written as

$$\nabla g(x) = \frac{\partial \phi(x, F(x))}{\partial a} + \nabla F(x) \frac{\partial \phi(x, F(x))}{\partial b}.$$

Hence we have

$$\langle \nabla g(x), d(x) \rangle = -\left\langle \frac{\partial \phi(x, F(x))}{\partial a}, \frac{\partial \phi(x, F(x))}{\partial b} \right\rangle - \left\langle \frac{\partial \phi(x, F(x))}{\partial b}, \nabla F(x) \frac{\partial \phi(x, F(x))}{\partial b} \right\rangle. \tag{18}$$

By property (P.2) shown in Lemma 5.1, the first term is nonpositive. Since $\nabla F(x)$ is positive semidefinite, the second term is also nonpositive. Hence we have

$$\langle \nabla g(x), d(x) \rangle \leq 0.$$

Assume now that $\langle \nabla g(x), d(x) \rangle = 0$. This implies the both terms of the right-hand side of (18) vanish. In particular, by the above-mentioned property (P.2), we have

$$\frac{\partial \phi_i(x_i, F_i(x))}{\partial a} \frac{\partial \phi_i(x_i, F_i(x))}{\partial b} = 0 \quad \text{for all } i = 1, \ldots, n.$$

It then follows from properties (P.1) and (P.3) in Lemma 5.1 that $\phi_i(x_i, F_i(x)) = 0$ for all $i$. Hence $x$ solves the NCP, a contradiction to the assumption. Therefore we must have $\langle \nabla g(x), d(x) \rangle < 0$, i.e., $d(x)$ is a descent direction of $g$ at $x$. □

Based on Lemma 6.1, we now present a derivative-free descent algorithm for solving the NCP.

**Algorithm 6.1** *(Descent method)*

*(S.0) Choose $x^0 \in \mathbb{R}^n$, $\varepsilon \geq 0$, $\sigma \in (0, 1/2)$, $\beta \in (0, 1)$ and set $k := 0$.*

*(S.1) If $g(x^k) \leq \varepsilon$, stop.*

*(S.2)* *Let* $d^k := -\dfrac{\partial\phi(x^k, F(x^k))}{\partial b}$.

*(S.3)* *Compute a steplength* $t_k = \beta^{m_k}$, *where* $m_k$ *is the smallest nonnegative integer satisfying the Armijo-type condition*

$$g(x^k + \beta^{m_k} d^k) \leq (1 - \sigma(\beta^{m_k})^2) g(x^k).$$

*(S.4)* *Set* $x^{k+1} := x^k + t_k d^k$, $k := k + 1$ *and go to (S.1).*

We summarize the convergence properties of Algorithm 6.1 in the following theorem. Here we suppose that $\varepsilon = 0$ in the algorithm and that the generated sequence is infinite.

**Theorem 6.1** *Let the NCP-functions* $\phi_i$ *be defined by (3) with* $\psi_0 \in \Psi_+^1$ *and* $\psi_i \in \Psi_{++}^2$ *for all* $i = 1, \ldots, n$, *and let* $g$ *be defined by (2). If* $F$ *is monotone, then Algorithm 6.1 is well-defined, and any accumulation point of the generated sequence* $\{x^k\}$ *is a solution of the NCP. Moreover, if the NCP is strictly feasible, then there exists at least one accumulation point.*

**Proof.** Since $F$ is monotone, $\nabla F(x)$ is positive semidefinite for any $x$. Hence, by Lemma 6.1, $d^k$ is a descent direction of $g$ at $x^k$ for each $k$. Then the first part of the theorem can be proved in exactly the same way as Proposition 11 in [21]. (In [21], Jiang considers a similar algorithm based on the squared Fischer-Burmeister function.) The last part follows from Theorem 4.1 and the descent property of Algorithm 6.1. □

Theorem 6.1 demonstrates in particular that the proposed algorithm is valid for the merit function $g$ constituted by the NCP-functions $\phi_{FB}^p$.

# 7. Final remarks

In this paper, we have introduced new merit functions for the nonlinear complementarity problem and proved several desirable properties of these functions, which compare favorably

with corresponding properties of other existing merit functions. On the other hand, one of the reasons why the squared Fischer-Burmeister function has attracted much attention of the researchers is the fact that it is the natural differentiable merit function for the following equation reformulation of the nonlinear complementarity problem:

$$\Phi_{FB}(x) := \begin{pmatrix} \phi_{FB}(x_1, F_1(x)) \\ \vdots \\ \phi_{FB}(x_n, F_n(x)) \end{pmatrix} = 0,$$

where $\phi_{FB}$ is the function displayed in Example 1.1. Similarly, we can view the merit functions presented in [14] and this paper as natural merit functions of the following two equations, respectively:

$$\Phi_{LT}(x) := \begin{pmatrix} \psi_0(\langle x, F(x)\rangle) \\ \psi_1(-x_1, -F_1(x)) \\ \vdots \\ \psi_n(-x_n, -F_n(x)) \end{pmatrix} = 0$$

and

$$\Phi(x) := \begin{pmatrix} \psi_0(x_1 F_1(x)) \\ \vdots \\ \psi_0(x_n F_n(x)) \\ \psi_1(-x_1, -F_1(x)) \\ \vdots \\ \psi_n(-x_n, -F_n(x)) \end{pmatrix} = 0,$$

where $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2, i = 1, \ldots, n$. Note that these are generalized nonlinear least squares reformulations of the nonlinear complementarity problem, and that there is no corresponding equation for the implicit Lagrangian by Mangasarian and Solodov [8].

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