# A New NCP-Function and Its Properties

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**Abstract.** Recently, Luo and Tseng proposed a class of merit functions for the nonlinear complementarity problem (NCP) and showed that it enjoys several interesting properties under some assumptions. In this paper, a similar idea to Luo and Tseng's, we present a new merit function for the NCP, which can be decomposed into component functions. We show that this merit function not only shares many properties with the one proposed by Luo and Tseng but also enjoys additional favorable properties owing to its decomposable structure.

**Key words:** Nonlinear complementarity problem, NCP-function, merit function, unconstrained optimization reformulation, error bound.

## 1. Introduction

The nonlinear complementarity problem (NCP) is to find $x \in R^n$ such that

$$[\text{NCP}] \qquad \langle x, F(x)\rangle = 0,\ x \in R^n_+,\ F(x) \in R^n_+,$$

where $F$ is a function from $R^n$ into itself, $\langle \cdot, \cdot \rangle$ denotes the inner product in $R^n$, and $R^n_+ := \{x \in R^n \mid x_i \geq 0, i = 1, \ldots, n\}$ [1, 2]. Recently, reformulation of the NCP as a minimization problem or a system of equations has drawn much attention [3]. A function which can constitute an equivalent minimization problem for the NCP is called a merit function. More precisely, a merit function is a function whose global minima on a set $X \subseteq R^n$ are coincident with the solutions of the original NCP. To construct such a function, it is effective to take advantage of the Cartesian structure of the NCP. In particular, the class of functions defined below serves as a convenient tool for constructing a merit function [4, 5, 6, 7].

**Definition 1.1** *A function $\psi : R^2 \to R$ is called an NCP-function if*

$$\psi(a, b) = 0 \Longleftrightarrow ab = 0, a \geq 0, b \geq 0.$$

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By using an NCP-function $\psi$, we can construct a merit function $g : R^n \to R$ for the NCP as follows: If $\psi$ is nonnegative on $R^2$, then define the function $g$ by

$$g(x) = \sum_{i=1}^{n} \psi(x_i, F_i(x)),$$

otherwise, by

$$g(x) = \sum_{i=1}^{n} \psi(x_i, F_i(x))^2.$$

It is easy to see that the NCP can be cast as the unconstrained optimization problem with the objective function $g$. There are many functions that belong to the class of NCP-functions. Among others, the following three NCP-functions have been well studied in the literature.

**(i)** $\psi_{NR}(a, b) = \min\{a, b\}$.

**(ii)** $\psi_{MS}(a, b) = ab + \frac{1}{2\alpha}(\max\{0, a - \alpha b\}^2 - a^2 + \max\{0, b - \alpha a\}^2 - b^2), \quad \alpha > 1.$

**(iii)** $\psi_{FB}(a, b) = \sqrt{a^2 + b^2} - a - b.$

The merit function based on the function $\psi_{NR}$ is called the natural residual, and it is not differentiable. The function $\psi_{MS}$ is nonnegative on $R^2$ [6] and the merit function based on $\psi_{MS}$ is the implicit Lagrangian proposed by Mangasarian and Solodov [8]. The function $\psi_{FB}$ was first considered by Fischer [9] (and attributed to Burmeister). Recently, the merit function based on $\psi_{FB}$ has extensively been studied and has shown to have a number of favorable properties. Among the above three NCP-functions, the following useful relations hold.

**Proposition 1.1** *[10, 11] There exist positive constants $c_1, c_2, c_3, c_4$ such that*

$$c_1\psi_{NR}(a, b)^2 \le \psi_{FB}(a, b)^2 \le c_2\psi_{NR}(a, b)^2$$

*and*

$$c_3\psi_{NR}(a, b)^2 \le \psi_{MS}(a, b) \le c_4\psi_{NR}(a, b)^2$$

*for all $a, b \in R$.*

This proposition indicates that if a merit function based on one of the three NCP-functions provides an error bound or has bounded level sets, then the same is true for the merit functions based on the other NCP-functions. To the authors' knowledge, for the merit functions based on these NCP-functions to provide global error bounds, not only the strong monotonicity (or the uniform $P$ property) but also the Lipschitz continuity of the function $F$ involved in the NCP are required [12]. The question whether or not the merit functions based on these NCP-functions provide global error bounds without the Lipschitz continuity of $F$ remains unanswered.

Recently, Luo and Tseng [13] proposed a new merit functions $f : R^n \to R$ defined by

$$f(x) = \psi_0(\langle x, F(x)\rangle) + \sum_{i=1}^{n} \psi_i(-x_i, -F_i(x)), \tag{1}$$

where $\psi_0 : R \to [0,\infty)$ and $\psi_i : R^2 \to [0,\infty), i = 1,\ldots,n$, are continuous functions that are zero on the positive orthant only. The function $f$ is not based on an NCP-function in the sense of Definition 1.1. Nevertheless, the function $f$ enjoys several interesting properties under some assumptions as shown in [13]. Among others, $f$ turns out to provide a global error bound under the strong monotonicity of $F$ only, and it is convex on $R^n$ under the assumption that $\langle x, F(x)\rangle$ and $-F_i(x), i = 1,\ldots,n$, are all convex functions of $x$

We may naturally come up with the idea of constructing a new NCP-function by using a similar approach to Luo and Tseng's. In view of the structure of the function $f$, we notice that the second term is decomposable into $n$ component functions but the first term is not. So, for constructing a new merit function based on an NCP-function, we may substitute the decomposable function $\sum_{i=1}^n \psi_0(x_i F_i(x))$ for the function $\psi_0(\langle x, F(x)\rangle)$. In other words, we consider the following $g : R^n \to R$:

$$g(x) = \sum_{i=1}^{n} \phi_i(x_i, F_i(x)), \tag{2}$$

where $\phi_i : R^2 \to R$ is defined by

$$\phi_i(a,b) = \psi_0(ab) + \psi_i(-a,-b) \tag{3}$$

for each $i = 1,\ldots,n$. It can be shown that $\phi_i$ is an NCP-function, and hence $g$ is a merit function for the NCP (see Lemma 2.1 and Theorem 2.1). Of course, the function $g$ is not the same as the merit function $f$. Nevertheless, we can expect that $g$ not only shares many properties with $f$ but also enjoys additional favorable properties owing to its decomposable structure. A possible cost to pay is that $g$ is in general nonconvex, while $f$ becomes convex when $F$ satisfies some conditions (see [13, Theorem 2.2]).

In this paper, we use the following concepts [14, 15].

**(a)** $F : R^n \to R^n$ is a $P_0$-function if

$$\max_{\substack{1\le i\le n\\ x_i\neq y_i}} (x_i - y_i)(F_i(x) - F_i(y)) \ge 0 \quad \text{for all } x, y \in R^n, x \neq y.$$

**(b)** $F : R^n \to R^n$ is a $P$-function if

$$\max_{1\le i\le n} (x_i - y_i)(F_i(x) - F_i(y)) > 0 \quad \text{for all } x, y \in R^n, x \neq y.$$

**(c)** $F : R^n \to R^n$ is a uniform $P$-function (with modulus $\mu$) if there is a $\mu > 0$ such that

$$\max_{1\le i\le n} (x_i - y_i)(F_i(x) - F_i(y)) \ge \mu\|x - y\|^2 \quad \text{for all } x, y \in R^n.$$

**(d)** An $n \times n$ matrix $M$ is a $P_0$-matrix if every principal minor of $M$ is nonnegative.

**(e)** An $n \times n$ matrix $M$ is a $P$-matrix if every principal minor of $M$ is positive.

It is known [15, Theorem 5.8] that if a $P_0$-function is differentiable, then its Jacobian is a $P_0$-matrix.

We shall use the following notation. For a vector $x \in R^n$ and an index set $\alpha \subseteq \{1,2,\ldots,n\}$, $x_\alpha$ represents the vector with elements $x_i, i \in \alpha$. We denote the (transposed) Jacobian of $F$ at $x$ by $\nabla F(x)$, i.e., $\nabla F(x)$ is the matrix whose $(i,j)$th element is $\partial F_j(x)/\partial x_i$. For a vector $t \in R^m$, $[t]_+$ denotes the vector whose $i$th element is $\max\{0, t_i\}$. We denote by $\mathrm{dist}(x, S)$ the distance between $x \in R^n$ and a set $S \subseteq R^n$.

## 2. Unconstrained optimization reformulation

In this section, we show that the function $\phi_i$ defined by (3) is an NCP-function, so that the unconstrained optimization problem constituted from this function is equivalent to the NCP. We also give conditions under which any stationary point of the merit function $g$ defined by (2) is a solution of the NCP.

To begin with, following [13], we define a class of functions, to which the component functions $\psi_i$ constituting $\phi_i$ are supposed to belong.

**Definition 2.1** *The class of continuous functions $\psi : R^m \to [0, \infty)$ such that*

$$\psi(t) = 0 \Longleftrightarrow t \leq 0$$

*is denoted by $\Psi^m$.*

As indicated by Luo and Tseng [13], an example of $\psi \in \Psi^m$ is

$$\psi(t) = \|[t]_+\|^2 \tag{4}$$

with $t \in R^m$ and an example of $\psi \in \Psi^2$ is

$$\psi(t) = \left( \left[ t_1 + t_2 - \sqrt{t_1^2 + t_2^2} \right]_+ \right)^2 \tag{5}$$

with $t \in R^2$.

The following lemma shows the most fundamental property of the function $\phi_i$ defined by (3).

**Lemma 2.1** *For any $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2$, the function $\phi_i$ defined by (3) is a nonnegative NCP-function.*

**Proof.** Since $\psi_0$ and $\psi_i$ are nonnegative by definition, $\phi_i$ is nonnegative. Now, we prove that $\phi_i$ is an NCP-function. First, suppose that $(a, b)$ is a vector such that $a \geq 0, b \geq 0$ and $ab = 0$. Then, by the definition of $\Psi^m$, we have $\psi_0(ab) = 0$ and $\psi_i(-a, -b) = 0$, which yields $\phi_i(a, b) = 0$. Next, suppose that $\phi_i(a, b) = 0$. Then, by the nonnegativity of $\psi_0$ and $\psi_i$, we must have $\psi_0(ab) = 0$ and $\psi_i(-a, -b) = 0$. Moreover, by the definition of $\Psi^m$, we obtain $ab \leq 0$, $a \geq 0$ and $b \geq 0$, from which it follows that $ab = 0$. Hence, $\phi_i$ is an NCP-function. □

In what follows, we suppose the function $\phi_i$ consists of functions $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2$. If $\psi_0$ and $\psi_i$ are differentiable, then $\phi_i$ is also differentiable and

$$\frac{\partial \phi_i(a, b)}{\partial a} = b\psi_0'(ab) - \frac{\partial \psi_i(-a, -b)}{\partial a}, \tag{6}$$

$$\frac{\partial \phi_i(a, b)}{\partial b} = a\psi_0'(ab) - \frac{\partial \psi_i(-a, -b)}{\partial b}. \tag{7}$$

Moreover, if $F$ is differentiable, then the function $g$ defined by (2) is also differentiable.

The following theorem shows that the unconstrained minimization problem

$$\min_{x \in R^n} \quad g(x)$$

is equivalent to the original NCP.

**Theorem 2.1** *The function $g$ defined by (2) is nonnegative on $R^n$. Moreover, $g(x) = 0$ if and only if $x$ is a solution of the NCP.*

**Proof.** By Lemma 2.1, $\phi_i$ is a nonnegative NCP-function for all $i = 1, \ldots, n$. Moreover, $g(x) = 0$ if and only if $\phi_i(x_i, F_i(x)) = 0$ for all $i = 1, \ldots, n$. Hence the theorem follows. □

Luo and Tseng [13] gave conditions under which the merit function $f$ defined by (1) is convex. However, the proposed merit function $g$ is not necessarily convex under such conditions. Nevertheless, since commonly used minimization methods can only find a stationary point or a local minimum, it is important to investigate conditions under which any stationary point of $g$ is a solution of the NCP. In the remainder of this section, we focus our attention to such conditions.

We define the following classes of continuous functions, which are given by Luo and Tseng [13].

**Definition 2.2** $\Psi_+^m$ *denotes the class of continuous functions $\psi \in \Psi^m$ such that*

**C.1** $\langle t, \nabla\psi(t)\rangle \geq 0$ *for all $t \in R^m$;*

**C.2** $\nabla\psi(t) \geq 0$ *for all $t \in R^m$;*

**C.3** $\nabla\psi(t) = 0 \Longleftrightarrow t \leq 0$.

*Moreover, $\Psi_{++}^m$ denotes the class of functions $\psi \in \Psi_+^m$ such that*

**C.4** $\nabla\psi(t) > 0$ *for all $t \not\leq 0$.*

The functions $\psi$ defined by (4) and (5) belong to $\Psi_+^m$ and $\Psi_+^2$, respectively. However, the function $\psi$ defined by (4) does not belong to $\Psi_{++}^m$, whereas the function $\psi$ defined by (5) belongs to $\Psi_{++}^2$.

We show some important properties of the NCP-function $\phi_i$.

**Lemma 2.2** *Let the NCP-function $\phi_i$ be defined by (3) with $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_{++}^2$. Then, $\phi_i$ has the following properties.*

**(P.1)** $\phi_i(a, b) = 0 \Longleftrightarrow \nabla\phi_i(a, b) = 0$;

**(P.2)** $\frac{\partial\phi_i(a,b)}{\partial a}\frac{\partial\phi_i(a,b)}{\partial b} \geq 0$ *for all $(a, b)^T \in R^2$.*

*Moreover, if $\psi_i \in \Psi_{++}^2$, then we have*

**(P.3)** $\frac{\partial\phi_i(a,b)}{\partial a} = 0 \Longleftrightarrow \frac{\partial\phi_i(a,b)}{\partial b} = 0$.

**Proof. (P.1)** First, suppose that $\phi_i(a, b) = 0$. Then, since $\phi_i$ is an NCP-function, we have

$$ab \leq 0, \quad \begin{pmatrix} -a \\ -b \end{pmatrix} \leq 0.$$

Hence, C.3 yields $\psi_0'(ab) = 0$ and $\nabla\psi_i(-a, -b) = 0$. It then follows from (6) and (7) that $\nabla\phi_i(a, b) = 0$. Next, suppose that $\nabla\phi_i(a, b) = 0$. Then, by (6) and (7), we obtain

$$b\psi_0'(ab) = \frac{\partial\psi_i(-a, -b)}{\partial a} \tag{8}$$

and

$$a\psi_0'(ab) \quad = \quad \frac{\partial \psi_i(-a,-b)}{\partial b}. \tag{9}$$

Let the both sides of (8) and (9) be nonzero. Then, since $\psi_0'(ab) \geq 0$ and $\nabla\psi_i(-a,-b) \geq 0$ by C.2, we must have $a > 0$ and $b > 0$ from (8) and (9). It follows from C.3 that $\nabla\psi_i(-a,-b) = 0$, which is a contradiction. Therefore, the both sides of at least one of (8) and (9) are zero. Without loss of generality, we assume the both sides of (8) are zero. Then, by C.3, we obtain $ab \leq 0$ or $b = 0$. It can easily be verified by considering the two case $ab \leq 0$ and $b = 0$ separately that the both sides of (9) are zero. Hence, we have $\nabla\psi_i(-a,-b) = 0$ and $\psi_0'(ab) = 0$. It follows from C.3 that $a \geq 0$, $b \geq 0$ and $ab \leq 0$, which yield $a \geq 0$, $b \geq 0$ and $ab = 0$. Since $\phi_i$ is an NCP-function, we obtain $\phi_i(a,b) = 0$.

**(P.2)** By C.1 and C.2, we have

$$ab\psi_0'(ab)^2 = (ab\psi_0'(ab))\,\psi_0'(ab) \geq 0. \tag{10}$$

We also have from C.2

$$\frac{\partial \psi_i(-a,-b)}{\partial a}\frac{\partial \psi_i(-a,-b)}{\partial b} \geq 0. \tag{11}$$

Furthermore, C.1 and C.2 yield

$$\begin{aligned} &\psi_0'(ab)\left(-a\frac{\partial \psi_i(-a,-b)}{\partial a} - b\frac{\partial \psi_i(-a,-b)}{\partial b}\right) \\ &= \quad \psi_0'(ab)\left\langle (-a,-b)^T, \nabla\psi_i(-a,-b)\right\rangle \geq 0. \end{aligned} \tag{12}$$

From (6) and (7), we have

$$\begin{aligned} \frac{\partial \phi_i(a,b)}{\partial a}\frac{\partial \phi_i(a,b)}{\partial b} \quad &= \quad ab\psi_0'(ab)^2 + \frac{\partial \psi_i(-a,-b)}{\partial a}\frac{\partial \psi_i(-a,-b)}{\partial b} \\ &\quad +\psi_0'(ab)\left(-a\frac{\partial \psi_i(-a,-b)}{\partial a} - b\frac{\partial \psi_i(-a,-b)}{\partial b}\right), \end{aligned} \tag{13}$$

which shows (P.2) by (10)-(12).

**(P.3)** Suppose that $\frac{\partial \phi_i(a,b)}{\partial a} = 0$. Then, by (13), we have

$$\begin{aligned} 0 \quad &= \quad ab\psi_0'(ab)^2 + \frac{\partial \psi_i(-a,-b)}{\partial a}\frac{\partial \psi_i(-a,-b)}{\partial b} \\ &\quad +\psi_0'(ab)\left(-a\frac{\partial \psi_i(-a,-b)}{\partial a} - b\frac{\partial \psi_i(-a,-b)}{\partial b}\right). \end{aligned}$$

Hence, by considering (10)-(12), every term of the right-hand side must be zero. C.3 and the first term being zero yield $ab \leq 0$. If either $a$ or $b$ is negative, then the second term is positive by C.4. This is a contradiction. So, we must have $a \geq 0$ and $b \geq 0$. Hence, we obtain $ab = 0$, $a \geq 0$ and $b \geq 0$, which yield $\phi_i(a,b) = 0$. Then, by (P.1) proven earlier, we have $\frac{\partial \phi_i(a,b)}{\partial b} = 0$. We can prove the converse in a similar way. □

Using Lemma 2.2, we establish the following theorem, which gives sufficient conditions under which any stationary point of $g$ solves the NCP.

**Theorem 2.2** *Let the NCP-functions $\phi_i, i = 1, \ldots, n$, be defined by (3) with $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_{++}^2$ (respectively, $\psi_i \in \Psi_+^2$), $i = 1, \ldots, n$, and let $g$ be defined by (2). If $F$ is a $P_0$-function (respectively, $\nabla F(x)$ is a $P$-matrix for any $x$), then any stationary point of $g$ is a solution of the NCP.*

**Proof.** We consider the case where $\phi_i, i = 1, \ldots, n$, are constituted by $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_{++}^2, i = 1, \ldots, n$. Suppose $F$ is a $P_0$-function. Then $\nabla F(x)$ is a $P_0$-matrix for all $x$. Let $x^*$ be a stationary point of $g$. Then we have

$$0 = \nabla g(x^*) = \sum_{i=1}^{n} \left( \frac{\partial \phi_i}{\partial a}(x_i^*, F_i(x^*))e_i + \frac{\partial \phi_i}{\partial b}(x_i^*, F_i(x^*))\nabla F_i(x^*) \right), \tag{14}$$

where $e_i$ is the $i$th column of the identity matrix. Let us denote the vectors $(\cdots, \frac{\partial \phi_i}{\partial a}(x_i^*, F_i(x^*)), \cdots)^T$ and $(\cdots, \frac{\partial \phi_i}{\partial b}(x_i^*, F_i(x^*)), \cdots)^T$ by $\frac{\partial \phi}{\partial a}(x^*, F(x^*))$ and $\frac{\partial \phi}{\partial b}(x^*, F(x^*))$, respectively. Then (14) can be rewritten as

$$0 = \frac{\partial \phi}{\partial a}(x^*, F(x^*)) + \nabla F(x^*)^T \frac{\partial \phi}{\partial b}(x^*, F(x^*)). \tag{15}$$

Let the index set $I(x^*)$ be defined by

$$I(x^*) = \left\{ i \;\middle|\; \frac{\partial \phi}{\partial b}(x_i^*, F_i(x^*)) \neq 0 \right\}.$$

Suppose that $I(x^*)$ is nonempty. Then, it follows from properties (P.1), (P.2) and (P.3) of $\phi_i$ that, for every $i \in I(x^*)$, both $\frac{\partial \phi_i}{\partial a}(x_i^*, F_i(x^*))$ and $\frac{\partial \phi_i}{\partial b}(x_i^*, F_i(x^*))$ are nonzero and have the same sign. Then for (15) to hold, it is necessary that $\nabla F(x^*)$ reverses the sign of $\frac{\partial \phi_i}{\partial b}(x_i^*, F_i(x^*))$ for all $i \in I(x^*)$. However, this contradicts the fact that $\nabla F(x^*)$ is a $P_0$-matrix [15, Theorem 5.8]. Therefore, $I(x^*)$ is empty, which means that $x^*$ is a solution of the NCP.

The case where $\phi_i, i = 1, \ldots, n$, are constituted by $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_+^2, i = 1, \ldots, n$, can be proved in a similar way. □

Next, we give a necessary and sufficient condition for a stationary point of $g$ to be a solution of the NCP. To this end, we define the index sets

$$\begin{aligned}
\mathcal{P}(x) &= \left\{ i \;\middle|\; \frac{\partial \phi_i(x_i, F_i(x))}{\partial b} > 0 \right\}, \\
\mathcal{C}(x) &= \left\{ i \;\middle|\; \frac{\partial \phi_i(x_i, F_i(x))}{\partial b} = 0 \right\}, \\
\mathcal{N}(x) &= \left\{ i \;\middle|\; \frac{\partial \phi_i(x_i, F_i(x))}{\partial b} < 0 \right\}, \\
\mathcal{R}(x) &= \mathcal{P}(x) \cup \mathcal{N}(x).
\end{aligned}$$

For simplicity, we shall denote these sets by $\mathcal{P}$, $\mathcal{C}$, $\mathcal{N}$ and $\mathcal{R}$. The point $x$ under consideration will always be clear from the context. Using these index sets, we define regularity conditions. Similar concepts of regularity have been considered by De Luca et al. [16], Facchinei and Kanzow [17] and Luo and Tseng [13] for different merit functions.

**Definition 2.3** *A point $x \in R^n$ is said to be regular if for every nonzero vector $z \in R^n$ such that*

$$z_{\mathcal{C}} = 0, \quad z_{\mathcal{P}} > 0, \quad z_{\mathcal{N}} < 0,$$

*there exists a vector* $y \in R^n$ *such that*

$$y_{\mathcal{P}} \geq 0, \;\; y_{\mathcal{N}} \leq 0, \;\; y_{\mathcal{R}} \neq 0$$

*and*

$$y^T \nabla F(x) z \geq 0.$$

*Moreover, a point* $x \in R^n$ *is said to be strictly regular if for every nonzero vector* $z \in R^n$ *such that*

$$z_{\mathcal{C}} = 0, \;\; z_{\mathcal{P}} > 0, \;\; z_{\mathcal{N}} < 0,$$

*there exists a vector* $y \in R^n$ *such that*

$$y_{\mathcal{C}} = 0, \;\; y_{\mathcal{P}} \geq 0, \;\; y_{\mathcal{N}} \leq 0$$

*and*

$$y^T \nabla F(x) z > 0.$$

The first regularity in Definition 2.3 is similar to that of [16] and holds particularly when $\nabla F(x)$ is a $P_0$-matrix. The second strict regularity is similar to that of [17], where it is simply called "regular". This strict regularity is satisfied if $\nabla F(x)$ is a $P$-matrix. However, when $\nabla F(x)$ is only positive semi-definite, $x$ is not necessarily strictly regular.

The following theorem shows that (strict) regularity is a necessary and sufficient condition for a stationary point of $g$ to be a solution of the NCP.

**Theorem 2.3** *Let the NCP-functions* $\phi_i, i = 1, \ldots, n$, *be defined by (3) with* $\psi_0 \in \Psi_+^1$ *and* $\psi_i \in \Psi_{++}^2$ *(respectively,* $\psi_i \in \Psi_+^2$*),* $i = 1, \ldots, n$, *and let* $g$ *be defined by (2). Suppose that* $x$ *is a stationary point of* $g$. *Then,* $x$ *is a solution of the NCP if and only if* $x$ *is regular (respectively, strictly regular).*

**Proof.** By using Lemma 2.2, it can be proven in a way similar to the proof of [16, Theorem 4.2] for the case where $\phi_i$ are constituted by $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_{++}^2$, and the proof of [17, Theorem 3.3] for the case where $\phi_i$ are constituted by $\psi_0 \in \Psi_+^1$ and $\psi_i \in \Psi_+^2$. □

Note that Theorem 2.2 can be obtained directly from Theorem 2.3.

## 3. Global and local error bounds

In this section, we give sufficient conditions on $F$, $\psi_0 \in \Psi^1$ and $\psi_1, \ldots, \psi_n \in \Psi^2$ for the merit function $g$ defined by (2) to provide a local or global error bound for the NCP. The approach taken here is parallel to that of Luo and Tseng [13].

First, let the merit functions $h : R^n \to R$ and $\hat{h} : R^n \to R$ be defined by

$$h(x) = \sum_{i=1}^{n} [x_i F_i(x)]_+ + \sum_{i=1}^{n} [-F_i(x)]_+ + \sum_{i=1}^{n} [-x_i]_+ \tag{16}$$

and

$$\hat{h}(x) = [\langle x, F(x) \rangle]_+ + \sum_{i=1}^{n} [-F_i(x)]_+ + \sum_{i=1}^{n} [-x_i]_+, \tag{17}$$

respectively. The functions $h$ and $\hat{h}$ are special cases of $g$ defined by (2) and $f$ defined by (1), respectively. The function $\hat{h}$ has been well studied in [18], and we will apply known conditions for $\hat{h}$ to provide an error bound to obtain conditions for $g$ to provide an error bound. Between the merit functions $h$ and $\hat{h}$, the following relation holds.

**Proposition 3.2** *Let $h$ and $\hat{h}$ be defined by (16) and (17), respectively. Then*

$$\hat{h}(x) \leq h(x) \quad \textit{for all } x \in R^n.$$

**Proof.** For all $x \in R^n$, we have

$$[\langle x, F(x)\rangle]_+ \leq \sum_{i=1}^{n} [x_i F_i(x)]_+.$$

The desired inequality then follows from the definitions of $h$ and $\hat{h}$. □

The error bound results established below rely on the following assumptions, which are related to growth behavior of functions $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2, i = 1, \ldots, n$. These concepts on growth property of a function have been introduced by Luo and Tseng [13].

**Assumption 1.** The functions $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2, i = 1, \ldots, n$, have local growth of order $q > 0$ in the sense that there exist positive constants $\tau$ and $\varepsilon$ such that, for each $i = 0, 1, \ldots, n$, $\psi_i$ satisfies

$$\psi_i(t) \geq \tau \|[t]_+\|^q \quad \text{for all } t \text{ with } \|t\| \leq \varepsilon.$$

**Assumption 2.** The functions $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2, i = 1, \ldots, n$, have asymptotic growth of order $q' > 0$ in the sense that there exist positive constants $\tau'$ and $\varepsilon'$ such that, for each $i = 0, 1, \ldots, n$, $\psi_i$ satisfies

$$\psi_i(t) \geq \tau' \|[t]_+\|^{q'} \quad \text{for all } t \text{ with } \|t\| \geq \varepsilon'.$$

The next proposition relates the function $g$ to the functions $h$ and $\hat{h}$.

**Proposition 3.3** *Let Assumption 1 be satisfied and the function $g$ be defined by (2). Then, for any $\rho > 0$, there exists a positive constant $\kappa$ such that*

$$\hat{h}(x) \leq h(x) \leq \kappa (g(x))^{1/q} \quad \textit{for all } x \textit{ with } \|x\| \leq \rho. \tag{18}$$

*If in addition Assumption 2 is satisfied, then there exists a positive constant $\kappa'$ such that*

$$\hat{h}(x) \leq h(x) \leq \kappa' \left( (g(x))^{1/q} + (g(x))^{1/q'} \right) \quad \textit{for all } x \in R^n. \tag{19}$$

**Proof.** The left inequalities of (18) and (19) follow from Proposition 3.2. The right inequalities of (18) and (19) can be proven in a way similar to [13, Lemma 3.1]. □

The inequalities (18) and (19) show that if $\hat{h}$ provides a global or local error bound, then $g$ also provides a global or local error bound.

We are now in a position to prove the main theorems of this section, which are parallel to Theorems 3.1 and 3.2 in Luo and Tseng [13].

**Theorem 3.1** *Suppose that each component function of $F$ is an analytic function. Also, suppose the solution set $S$ of the NCP is nonempty. Let Assumption 1 be satisfied and the function $g$ be defined by (2). Then, for any $\rho > 0$, there exist positive constants $\tau$ and $\gamma$ such that*

$$dist(x, S) \leq \tau(g(x))^{\gamma/q} \quad \text{for all } x \text{ with } \|x\| \leq \rho.$$

*Moreover, if $F$ is affine, the inequality holds with $\gamma = \frac{1}{2}$.*

**Proof.** This theorem directly follows from Proposition 3.3 and [18, Theorem 5.3]. □

**Theorem 3.2** *Suppose that $F$ is a uniform $P$-function with modulus $\mu$. Let Assumptions 1 and 2 be satisfied and the function $g$ be defined by (2). Then there exists a positive constant $\tau$ such that*

$$\|x - x^*\|^2 \leq \tau \left( (g(x))^{1/q} + (g(x))^{1/q'} \right) \quad \text{for all } x \in R^n,$$

*where $x^*$ is the unique solution of the NCP.*

**Proof.** For any $x \in R^n$, by the uniform $P$-property of $F$, there exists an index $i$ such that

$$\begin{aligned}
\mu\|x - x^*\|^2 &\leq (x_i - x_i^*)(F_i(x) - F_i(x^*)) \\
&= x_i F_i(x) - x_i^* F_i(x) - x_i F_i(x^*) \\
&\leq [x_i F_i(x)]_+ + x_i^*[-F_i(x)]_+ + F_i(x^*)[-x_i]_+ \\
&\leq \tau' \left( [x_i F_i(x)]_+ + [-F_i(x)]_+ + [-x_i]_+ \right) \\
&\leq \tau' \left( \sum_{i=1}^n [x_i F_i(x)]_+ + \sum_{i=1}^n [-F_i(x)]_+ + \sum_{i=1}^n [-x_i]_+ \right) \\
&= \tau' h(x),
\end{aligned}$$

where $\tau' = \max\{1, x_i^*, F_i(x^*)\} > 0$ and the first equality follows from the fact $x_i^* F_i(x^*) = 0, i = 1, \ldots, n$. The desired inequality then follows from Proposition 3.3. □

Note that Theorem 3.2 in [13] assumes that $F$ is strongly monotone, whereas Theorem 3.2 here assumes that $F$ is a uniform $P$-function.

Now we turn our attention to the boundedness of level sets of the merit function $g$. This property is useful in ensuring a descent method of minimizing $g$ to generate a bounded sequence. Note that by Theorems 3.2, we can immediately give conditions under which $g$ has bounded level sets. It is possible, however, to give more general conditions for the function $g$ to have bounded level sets, as shown in the next theorem. This result is a straightforward modification of the corresponding result by Luo and Tseng [13, Theorem 4.1].

**Theorem 3.3** *Suppose that, for any sequence $\{x^k\}$ such that $\|x^k\| \to \infty$, $[-x^k]_+/\|x^k\| \to 0$ and $[-F(x^k)]_+/\|x^k\| \to 0$,*

$$\liminf_{k \to \infty} \frac{\max_i \{x_i^k F_i(x^k)\}}{\|x^k\|} > 0.$$

*Let $\psi_0 \in \Psi^1$ and $\psi_i \in \Psi^2, i = 1, \ldots, n$, have the property that $\|[t]_+\| \to \infty$ implies $\psi_i(t) \to \infty$ for $i = 0, 1, \ldots, n$. Then the function $g$ defined by (2) has bounded level sets.*

**Proof.** The theorem can be proven in a way similar to [13, Theorem 4.1]. □

We can easy verify that

$$\liminf_{k\to\infty} \frac{\langle x^k, F(x^k)\rangle}{\|x^k\|} > 0 \implies \liminf_{k\to\infty} \frac{\max_i\{x_i^k F_i(x^k)\}}{\|x^k\|} > 0.$$

So, the assumptions on $F$ in Theorem 3.3 are weaker than those in [13, Theorem 4.1], which hold, as pointed out in [13], whenever $F$ is a uniform $P$-function or $F(x) = Mx + q$ with $M$ being an $R_0$-matrix.

## 4. Concluding remarks

In this paper, we have considered a new NCP-function and shown that this function constitutes a merit function for the NCP. We have given some conditions under which any stationary point of the merit function is a solution of the NCP. We have also given some conditions for the merit function to provide error bounds for the NCP.

In [5, 13, 19, 20], descent methods which do not require the evaluation of $\nabla F(x)$ have been proposed for different merit functions for the NCP. For the merit function $g$ considered in this paper, it can be shown that the vector $\frac{\partial \phi}{\partial b}(x, F(x))$ defined in the proof of Theorem 2.2 is a descent direction of $g$ at $x$ if $\nabla F(x)$ is positive semi-definite. So, we may develop a descent method which uses the vector $\frac{\partial \phi}{\partial b}(x, F(x))$ as a search direction.

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