Machine Formulation and Understainding of the Equations of Physical Laws by Symmetry

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Abstract

Based on the view that symmetry plays an essential role in human reasoning about the laws of physical phenomena, we propose a reasoning paradigm in which symmetry assists in the reasoning about equations physical laws. Within this paradigm, symmetries are used as constraints which enable us to specify, derive and generalize these equations. The symmetry-based reasoning is extracted and formalized from Einstein's work on relativity. We claim that the reasoning procedure thus formalized provides a general reasoning architecture that is common to dimensional analysis in engineering, mathematical proofs, and commonsense reasoning. This symmetry-based reasoning system has been implemented as a symbol-processing system with a production system and a formula-processing system. Using the symmetry-based reasoning system, the equation of Black's law of specific heat is demonstrated to be specified.

Key Words - Machine discovery and formulation, qualitative physics, scientific discovery, common sense reasoning, symbolic Computation

1. Introduction

One aspect of symmetry that has not yet been fully addressed is its function in human reasoning. Despite the fact environment is full of symmetry and that humans depend symmetry for perception, memory and reasoning, relatively little work on symmetry has been done in artificial intelligence. Leyton's work on the role of symmetry in reasoning about shape [Leyton, 88] and Liu and Popplestone's work on spatial reasoning [Liu and Popplestone, 90] are exceptions to this general trend.

The ultimate goal of our research is to introduce a reasoning that uses symmetry to enhance the intelligence of the system. In this paper, we focus on a specific reasoning based on symmetry found in physics. We formalize the reasoning that we can trace in Einstein's 1905 paper [Einstein, 05] which emphasizes that the physical law should be expressed in a way invariant under reference frame. Based on this principle, we specify and derive the equations of physical laws by using symmetry as a guiding constraint.

The explicit investigation of symmetry-based reasoning in physics is relatively modern. Such reasoning is evident in the special and general

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relativity theory, the theory of quantum mechanics, the theory of elementary particles, and superstring theory. Early attempts to relate symmetries in physical laws to conservation laws can be seen in the 1918 work of Noether or even the older work by Hamel in 1904. Nevertheless, the origin of this investigation may be traced back to Newtonian physics, which implemented symmetries such as space translation, time translation, time reversal and parity.

Further, this investigation is not restricted to physics. If we take a broad definition of symmetry, such common sense reasoning as "if it takes one hour to go from A to B, then it would take also one hour from B to A," can be recognized as examples of symmetry-based reasoning. In general, symmetry-based reasoning seems to be a weak method to which we resort when powerful knowledge such as the dynamics of a particular system is not readily available, as occurs with the theory of elementary particles.

Viewed as a reasoning system, the symmetry-based reasoning is distinct in that it is reasoning about *modeling* rather than reasoning about the *behavior* of given models. Although reasoning about behavior has been widely discussed and implemented -- primarily in artificial intelligence -- little has been done on the important topic of modeling.

Viewed as a law-discovery system [Simon, 77], our implementation is in the same line as that of **Black System** [Langley *et al.*, 87] in that it is theory-driven and uses a conservation concept. Although **Bacon 5** [Langley *et al.*, 81] uses symmetry in part to economize search, the implementation here relied on symmetry to form the base of the reasoning paradigm.

Although we address symmetry-based specification and derivation of the equations of physical laws, we are fully aware not only that the final justification of the law should be against experimental data but that experimental data are required to bridge the gap between physical laws and symmetry.

Applying within a specific *symbol system* [Simon, 69] would depend on the system's ability to identify symmetry in the representation adopted. Currently, it is possible for a computer symbol system to identify symmetry to some extent in symbolically expressed formulae, -- something which, up to now has been possible only for a human symbol system using paper and pencil.

The literature of physics includes several principles [Bridgman, 62, Feynman, 65, Wigner, 67, Rosen, 83, Brandmuller, 86, van Fraassen, 89, Froggatt, 91], which allow us to use symmetry in reasoning. The principle we used in order to specify and derive the equations of physical laws is *Symmetry as Guiding Constraints*. At this point, we need to formally state what we mean by symmetry. The following definition is merely a restatement of that by Weyl [Weyl, 52]: An object O has symmetry under a mapping T that can operate on O if T(O) = O. We use the word transformation when the mapping is defined on the symbolic formulae.

To illustrate the use of these principles we will consider the problem of specifying figures by rotational symmetry around a center of rotation on a plane. Let a symmetry defined by a rotation of $2\pi/n$ around the center point be denoted by C_n. As an example of using *Symmetry as Guiding Constraints,* consider the problem of specifying figures by the constraint of C_6 rotational symmetry. Obviously, this symmetry cannot completely specify the object as a hexagon, although it can eliminate the possibility of pentagon or octagon.

Section 2 discusses the extraction of reasoning with symmetry from Einsteins's work on relativity. Derivation of the formula of Pythagorean theorem and dimensional analysis are discussed as one of the symmetry-based reasoning. Section 3 presents primitives of symmetry-based reasoning. In section 4, the symmetry-based reasoning is formalized as a symmetry-based specification and derivation, and the example of specifying the equations of Black's law is presented.

2. Overview of Symmetry-Based Reasoning

2.1. Extraction of Symmetry-Based Reasoning from Einstein's Work

This work is motivated by Einstein's work on relativity: Einstein's reasoning can be formalized as symmetry-based reasoning that fully uses symmetry as a constraint, and the reasoning can be applied not only to motion but also to other physical reasoning or even common sense reasoning which is applied to the domain other than physics.

Hence, we first discuss Einstein's work of relativity and formalize the reasoning paradigm as symmetry-based reasoning.

Then, as an application to artificial intelligence, the symmetry-based reasoning can be implemented as deriving the symbolic forms with the symbolic rewriting system.

When one reviews Einstein's work on special and general relativity from the viewpoint of reasoning paradigm, one would realize they have a reasoning style in common: Both special and general theory of relativity have two components, *i.e.* , the principle of relativity and the entity carrying physical meaning. As will be discussed in section 4 in the more general context of symmetry-based reasoning, we identify the former as symmetries and the latter as object. In order to derive the symbolic forms either for the mapping defined in symmetries or for the object, symmetries are used as constraints that the object must satisfy.

Specifically, the principle of relativity (as in special theory of relativity) states that the physical law should be invariant when viewed from different coordinate systems; one moves with a constant velocity relative to the other. The application of the symmetry (special relativity) to the object (constancy of the speed of light) leads to the following: The light speed must be invariant viewed from different coordinate systems. This fact is used to derive the Lorentz transformation, which correspond to the mapping defining symmetry. Afterwards, the Maxwell-Hertz equations are checked against the Lorentz transformation to see whether or not the equations are invariant under the transformation. Further, the same style of symmetry-based reasoning is used to generalize the equations for the composition of velocity and those for the Doppler effect. In this reasoning, symmetry is the same as the above (i.e., special relativity), but it applied to the object different from the constancy of the speed of light.

In summary, symmetry-based reasoning, as demonstrated by Einstein's work on relativity, has two components: symmetry as shown in the principle of relativity (in the special theory of relativity) and objects carrying physical meaning as in the constancy of the speed light (in the special theory of relativity). The symmetry-based reasoning is carried out by using the symmetry as constraint to specify, derive, and modify the object. In the next subsection, we will show an example which use symmetry-based reasoning and which is outside of the physics domain. This will show the symmetry-based reasoning (as extracted from Einstin's work) is so general that it can be applied to physics as well as to other areas such as geometry and modeling in engineering.

2.2. Geometric reasoning as a symmetry-based reasoning

It is interesting that the same reasoning as that explored in relativity theory can be applied to many fields. One of the fields is geometry.

Example 2.2. (specifying the formula of Pythagorean theorem)

An important symmetry for geometric objects is that the interrelation among the components of geometric objects is invariant under dilation(expansion of the size). In other words, what defines the geometric object is not the absolute length of the components but the relation of the lengths among them. For example: for the right triangle shown in Figure 1, the relation between the length of the three edges must be invariant under dilation. The relation, thus, should be represented in a symbolic form which is invariant under dilation. This is completely parallel to the reasoning in Einstein's relativity that the symbolic form of the physical law should be invariant under the different frame of reference. In our context of symmetry-based reasoning, the following two components are given.

O (Object to be specified): c = f(a,b).

T (Transformation of the symmetry): c-> kc, a-> ka, b-> kb where k is any real number and v_i -> g(v_j) denotes a substitution.

Thus, O =T(O) gives the constraint: k f(a,b) = f(ka, kb) that is used to specify the symbolic form of f(a,b). For three right triangles, the big one and two internal ones, equivalent forms must hold: c = f(a,b), a = f(a^2/c, ab/c), b= f(ab/c, b^2/c).□ The length c can be obtained by adding the two edges from the two internal right triangles; c = a^2/c + b^2/c. Thus, separating c into the left hand side yields: c = $\sqrt{a^2+b^2}$.

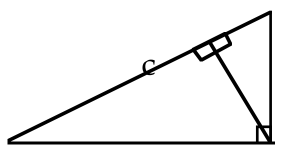

Figure 1 Geometric object for Pythagorean theorem

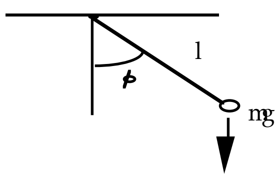

Figure 2 A simple pendulum where the particle with mass m is suspended by a string

2.3. Symmetry-based reasoning as a generalization of dimensional analysis

One of the important difference between geometry and physics can be found in the scale symmetry. In physics the scale symmetry does not hold (as pointed out, for example, by [Feynman 65]): i.e., if you make a miniature system whose size is, say one tenth of the real system, then you cannot expect everything is same for the miniature as the real one even though everything is made same as the original one except size (known as scale effect). However, the scale symmetry for unit of measure of independent dimension must hold for physical system as known from Buckingham's $\prod$ theorem [Buckingham 14] [1].

Again, it should be noted that this Buckingham's $\prod$ theorem is understood as the result of applying same reasoning as Einstein's theory of relativity: the representation of physical law (the equation describing a physical system in this case) must be represented as a form invariant under change of frame of reference (the system of dimension in this case). In fact, $\phi(\pi_1, \pi_2, \ldots \pi_{n-r})$ is such an invariant form since $\pi_1, \pi_2, \ldots \pi_{n-r}$ are invariant under scale change of unit of measure for each independent dimension.

The next example illustrates not only that the dimensional analysis can be done within the framework of symmetry-based reasoning but that the symmetry of scale change of unit of measure for independent dimension in dimensional analysis can be treated as constraints in symmetry-based reasoning, similarly to the other symmetries.

Example 2.3. (Dimensional analysis) [Sedov 43]

Consider the motion of a simple pendulum where the particle with mass m is suspended by a string as shown in Figure 2.

The object to be specified is the equation of the system f(t, l, g, m, ø) where t: time(T), l: length of string(L), g: gravity constant (L/(T^2)), m: mass of the pendulum(M), and ø : angle of the string (dimension is indicated inside the parentis).

The given symmetries are:

- (1) scale symmetries due to the basic dimensions T, L, M.,
- (2) phase translatory symmetry in terms of time t, and
- (3) mirror symmetry in terms of the angle ø.

The symmetry (1) specify the form f(t, l, g, m, ø) to: f($t\sqrt{g/l}$, ø). The symmetry (2) further gives the constraint:f($t\sqrt{g/l}$,ø)=f($(t+Tc)\sqrt{g/l}$, ø). The symmetry (3) further gives the constraint: f($t\sqrt{g/l}$, ø) = f($t\sqrt{g/l}$, -ø).

Our method of specifying the equations of physical law can be viewed as a generalization of that in dimensional analysis. Table 1 summarizes the parallelism of the symmetry-based reasoning among Einstein's theory

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[1] If a physical system is described by f($x_1, x_2, \ldots, x_n$) where $x_1, x_2, \ldots, x_n$ are n variables that involve r basic dimensions, then the physical system will be described only by n-r independent dimensionless products $\pi_1, \pi_2, \ldots \pi_{n-r}$. The equation describing the physical system is, thus, reduced to be $\phi(\pi_1, \pi_2, \ldots \pi_{n-r})$.

of relativity, derivation of the form of Pythagorean theorem and dimensional analysis.

Table 1 Parallelism of the symmetry-based reasoning among Einstein's theory of relativity, derivation of the form of Pythagorean theorem and dimensional analysis.

	Einstein's theory of relativity	Geometric reasoning in Pythagorean theorem	Dimensional analysis
Symmetries	Symmetry with frame of reference	Scale symmetry	Scale symmetry for unit of measure of independent dimension
Object with the Content of the Domain	Constancy of the speed of light	Geometric Configuration	Involved variables and their physical dimensions

On the one hand, insight of what symmetries are involved in physical phenomena or mathematical statement is important element. Using these symmetries in deriving and specifying the formulae by symbolic processing is a core of our symmetry-based reasoning.

On the other hand, it should be noted that finding the object with domain specific content is an important core in scientific discovery. However, this process of finding the object with content is far from automating or even from formalizing at this stage.

As will be seen in section 4.2., if the given object is specific enough, the symmetry-based reasoner carry out deductive reasoning. However, if the given object is not specific, the reasoner carry out inductive reasoning by proposing candidate forms satisfying the given symmetry and modifying the current forms to meet the other given symmetries.

3. Primitives of Symmetry-Based Reasoning

3.1. Fixed point and symmetry

Our method of specifying the equations of physical laws is a generalization of that in dimensional analysis. Buckingham's $\prod$ theorem can be generalized along the following lines (where the symmetry used is scale symmetry for unit of measure of an independent dimension and the invariant form is a dimensionless product in the Buckingham theorem):

If a equation is to be described with variables: $x_1, x_2, \ldots, x_n$ and if the equation should have a symmetry that can be attained only by a form F($x_1, x_2, \ldots, x_n$), then the target equation can be reduced to the form f(F($x_1, x_2, \ldots, x_n$)).

If we have a unique form like this that attains symmetry, then our task is to modify the form by applying the commutative operators so that it will satisfy the other symmetries or physical requirements, such as dimensional agreement. However, it is difficult to find such a unique form or to guarantee its uniqueness under circumstances lacking a theorem corresponding to the fixed point theorem.

Example 3.1.1.

If the given symmetry is the translatory symmetry such that the form should be invariant under the translation: $x_1, x_2, \ldots, x_n$ -> $x_1 + c, x_2 + c, \ldots, x_n + c$ where c is an arbitrary real number, then the unique

[2] form is: F($x_1, x_2, ..., x_n$) = $a_1x_1 + a_2x_2 +, ..., +a_nx_n$ where $a_1, a_2, ..., a_n$ are constants satisfying $a_1 + a_2 +, ..., +a_n = 0$. Therefore, the equation with this translatory symmetry can be reduced to the form f(F($x_1, x_2, ..., x_n$)) by the above argument. (This form will be used in the example 4.2.1. to specify Black's law.) However, if the given symmetry is the permutation symmetry: x_i <-> x_j (v_i <-> v_j denotes a permutation), then there are many forms satisfying the symmetry. Examples are: F(x_i, x_j) = $x_i x_j$, F(x_i, x_j) = $x_i + x_j$ and F(x_i, x_j) = $x_i x_j + x_i + x_j$.

The process of specifying the equations of physical laws is a process of finding an object (formula) whose symmetry is as close as possible to the strength of the symmetries given as constraints. Due to the absence of symbolic version of the fixed point theorem, specification and derivation process is driven by observing how T(O) differs from O and by using the equation O = T(O). There is no continuous measure that indicates how close the current form is to the target solution in the search for the symbolic form in our procedure of symmetry-based reasoning. The discrete measure indicating how close the current form is to the target solution is the number of how many symmetries (out of the symmetries given as the constraints) the current form satisfies. One heuristic to increase the efficiency of the search is that if the current solution satisfies the symmetry, then all the symmetries weaker than the symmetry can be disregarded.

Example 3.1.2

Since a regular triangle already has the symmetry C_3, this symmetry does not provide any information to the regular triangle. However, both the symmetries C_4 and C_6 will provide the information, for they require the figure to be a regular 12-gon and a hexagon, respectively. C_6 is stronger symmetry than C_3 because C_3 cannot provide information for all the objects to which C_6 can provide information. C_4 is orthogonal to C_3 because C_3 can provide information for some objects to which C_4 cannot provide information. In this sense, the continuous symmetry C_∞, which specifies a circle, is the strongest possible symmetry.

In section 4.2., we will look at Black's law of specific heat as an example of symmetry in a physical law.

3.2. Formula-processing for symmetry-based reasoning

The formula-level operations required for the symmetry-based reasoning are: symmetry identification, equation building by symmetry, and equation solving. In our study, we used *Mathematica* [Wolfram, 88] for formula-processing. The intrinsic problem in symmetry identification comes from the lack of an ultimate canonical form of the equations, which makes it difficult for the system to identify two formulae as equivalent. Thus, even if the two forms are not identified as equivalent by the system, it may be simply because the system cannot identify the equivalence.

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[2]We do not derive the uniqueness here because the derivation process itself is irrelevant with the symmetry-based reasoning; we use the heuristics associating the given symmetry with possible forms satisfying the symmetry, as seen in the following sections.

We used a production system to implement the symmetry-based reasoning. The reasons we adopted the production system architecture are twofold: First, we formalized symmetry-based reasoning as based on a human reasoning paradigm rather than a computational algorithm. Second, symmetry-based reasoning requires much heuristic knowledge which is not explicitly included in the formula.

We implemented a production system on *Mathematica* so that the formula-level operations mentioned above could be done within the production system. We present the syntax and examples of each of these components in the following subsections.

3.2.1. Transformation of equations

The syntax of transforming an expression is: `Trans_[exp_,{parameter-list}]`. There are five types of transformations commonly used: **Transl, Perm, Dilat, Com, ASCom**. The symmetry defined by the transformation will be expressed as follows: `Trans_{parameter-list}`.

- **Transl[exp_,{x1,x2,...},c]** will translate the listed variables x1,x2,... to x1+c ,x2+c ,... in the expression specified in exp_.
- **Perm[exp_,{x1,x2,y1,y2...}]** will permute x1 with x2, y1 with y2, and so on.
- **Dilat[exp_,{x1,x2,...},c]** will dilate the listed variables x1,x2,... into c□x1 ,c x2 ,....
- **Com[exp_,{x1,x2}]** will permute x1 with x2.
- **ASCom[f_[x1,x2],{x1,x2}]** will make the function f_[x1,x2] into 1 - f_[1/x1,1/x2].

Other transformations which are specific to the problems will be described at each example. In the examples of the rest of paper, the input to the system is in boldface. The output from the system is not.

Example 3.2.1

```
Transl[-(c1 m1 t1) - (c2 m2 t2) + cf (m1 + m2)
tf,{tf,t1,t2},c]
-(c1 m1 (c + t1)) - c2 m2 (c + t2) + cf (m1 + m2) (c + tf)
ASCom[f[m2/m1,c2/c1],{m2/m1,c2/c1}]
      m1  c1
1 - f[--, --]
      m2  c2
```

3.2.2. Symmetry identification

The symmetry of the equation can be mathematically identified by investigating the equivalence between the original equation and the equation after transformed as described above. The syntax of the symmetry identification is:

`SymQ[{},exp_,Trans_,,{parameter-list}]` which will identify the expression `exp_` is equal to the expression after the transformation: `Trans_[exp_,{parameter-list}]`. It will return `True` if the equality is identified, otherwise it returns `False`. As we have mentioned, it may return `False` even if the expression and that after complicated reduction are equal.

Example 3.2.2

```
SymQ[{},-(1 - (1 + (c2 m2)/(c1 m1))^(-1) + t2/((1 + (c2
m2)/(c1 m1)) t1)) + tf/t1,Perm,{m1,c1,m2,c2}]
```

```
True
SymQ[{},f[c1, c2, m1, m2, tf, t1, t2],Transl, {tf, t1, t2}]
False
```

3.2.3. Proposing formula by symmetry

Symmetries can be used to propose possible forms that the target object may have. Since such proposals are made based on heuristics, the resulting object must be evaluated (see section 4.1. for the process) to see whether if they satisfy all the given symmetries. The syntax of proposing possible forms is: `Trans_[prop,exp_,{parameter-list}]`.

Example 3.2.3

```
Transl[prop,f[c1, c2, m1, m2, tf, t1, t2],{tf, t1, t2}]
 tf - (t1 (1 - f[c1, c2, m1, m2]) + t2 f[c1, c2, m1, m2])
Dilat[prop,tf - (t1 (1 - f[c1, c2, m1, m2]) + t2 f[c1, c2,
m1, m2]),{c1, c2}]
                      c2                          c2
 tf - (t1 (1 - f[1, --, m1, m2]) + t2 f[1, --, m1, m2])
                      c1                          c1
```

3.2.4. Symmetry derivation from constraint

Symmetries on the equation such as permutation symmetry can provide a constraint which can be used to derive internal symmetries (symmetry on the part of the original equation), as seen in the example of symmetry-based specification of the equation of Black's law. To carry out a symmetry derivation,

- First, eliminate irrelevant variables from the constraint (given by the symmetry).
- Then, use pattern-matching to search the symmetry for the extracted part of the original equation.
- After the internal symmetries are detected, they are recast to the symmetry-based reasoning system by creating a subgoal of specifying the part of the original equation by the internal symmetries detected.

4. Symmetry-based reasoning

4.1. Procedure of symmetry-based reasoning

In his 1905 paper [Einstein, 05], Einstein used the symmetry-based reasoning. It should be noted that he used this reasoning to derive symmetry of the Lorentz transformation [Lorentz, 04] and to derive objects[3]. By extracting the reasoning from Einstein's work and making it to fit as a procedure, the symmetry-based reasoning may be formalized as:

- ***(step 0) Given: transformation T and some piece of information for an object O.***
- ***(step 1) Checking symmetry: if the object is given in symbolic form, then check whether the object satisfies the given symmetry. If O = T(O), stop; otherwise go to (step 3)***
- ***(step 2) Proposing object: If object O does symmetry, then propose a candidate formula of the object O by heuristics on the basis of the given list of symmetries T.***
- ***(step 3) Modifying object:***
  - ***(step 3.1) if O = T(O) can derive new transformation T′ for the part of object O′ then go back to (step 0) with these T′ and O′.***

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[3]The generalized equation of Doppler's principle, the equation of pressure of light, and the equation for a relativistic mass[Einstein, 05].

- ***(step 3.2) if O = T(O) include unknown parameters, solve the equation O = T(O).***
- ***(step 3.3) modify O by heuristics so that O = T(O) is satisfied. If there is not enough knowledge to do this, go back to (step 2) for new proposal.***

The rules of the symmetry-based reasoner can be divided into the procedual steps above. The rules of the step `1` see if given symmetries are satisfied by the given object using `SymQ` described above. If all the given symmetries are satisfied by the object, then the rules of this step simply terminate the task. A simple symmetry check can be done within this step. We call this mode *symmetry identification*.

If any of the symmetries are not satisfied by the object and if there is any knowledge available to propose modifications to the object so that the unsatisfied symmetry may be satisfied after the modification, then the rules of the step `2` will make those modifications. Among several symmetries which can propose modifications, permutation symmetries work in a unique manner. Permutation symmetries do not propose the modification, but rather derive constraints at the step `3.1` and then derive the internal symmetries found in this step. Next, the permutation symmetries create a subgoal of specifying the part of the object using the new symmetries found at this step. After these modifications and after working memory elements have been preprocessed, the object is recast to the symmetry check process returning the control to the step `1`. We call this mode *symmetry-based specification*.

If the modification is just a specification of unknown parameters in the object, then the rules of the step `3.2` are evoked after the step `1`. In this step, the given symmetries are used as building the equation whose solution will specify the unknown parameter. The modification on the object is made by substituting the solution to the unknown parameter at the next step `3.3`. After this modification, the same process as that of *symmetry-based specification* follows. We call this mode *symmetry-based derivation*. In the following, we present sample sessions for these three modes.

4.2. Examples of symmetry-based reasoning

In this section, we will give only example of *symmetry-based specification,* since *symmetry-based derivation* and *symmetry identification* use only a part of steps of *symmetry-based specification*. The examples of *symmetry identification* mode tested in our reasoner include that the angular momentum is invariant under spatial rotation, that the quadratic form of $-(c^2t^2) + x^2 + y^2 + z^2$ is invariant under the Lorentz transformation, and that the Maxwell equation is invariant under the Lorentz transformation. The examples of *symmetry derivation* of parameters include that the specific heat from the symmetries used in the next example 4.2.1 when given object has only specific heat left as unspecified parameter, the parameters in Lorentz transformation, the angle between spin and velocity for mass-less particle is zero[4], which is more complex than those shown here.

Example 4.2.1 (Specification of the equation of Black's Law)

Black's law of specific heat can be stated as follows: If two entities, whose initial state described by temperature t_1 and t_2, specific heat c_1 and c_2, and mass m_1 and m_2, are thermally coupled, then the final equilibrium temperature will be $t_f = (c_1 m_1 t_1 + c_2 m_2 t_2)/(c_1 m_1 + c_2 m_2)$. In the following example, the equation of Black's law: $t_f = (c_1 m_1 t_1 + c_2 m_2 t_2)/(c_1 m_1 + c_2 m_2)$ will be specified starting from scale symmetry (dimensional analysis), translatory symmetry of temperature, permutation symmetry of interacting entities, and permutation symmetry of specific heat and mass. This example specifies the equations of Black's Law when sufficient symmetries are given. Both the input and the output have been translated to higher-level expression for readability. Also, some inessential parts are omitted. The step numbers in parentheses refer to the procedural step described in the previous subsection. The number before parentheses indicates the depth as well as the cycle of the procedure. `==`, `!=` and `&&` denote equality, inequality and logical and, respectively.

```
O:  f[c1,c2,m1,m2,tf,t1,t2]
T:  Perm{m1,m2,c1,c2,t1,t2},Perm{m1,c1,m2,c2},  Dilat{m1,m2},
Dilat{c1,c2},  Dilat{t1,t2,tf},  Transl{tf,  t1,  t2}
1(step 1.1) The system identifies that the symmetry
Transl{tf, t1, t2} is not satisfied by the object.
```

. . . . symmetry checking given in the list T, similar to the above. . . . .

```
1(step 2) By the symmmetry Transl{tf, t1, t2}, the system proposed
the modification from the object:
 f[c1, c2, m1, m2, tf, t1, t2]
to:
 tf - (t1 (1 - f[c1, c2, m1, m2]) + t2 f[c1, c2, m1, m2])
1(step 2) By the symmmetry Dilat{c1, c2}, the system proposed the
modification from the object:
tf - (t1 (1 - f[c1, c2, m1, m2]) + t2 f[c1, c2, m1, m2])
to:                c2                      c2
tf - (t1 (1 - f[1, --, m1, m2]) + t2 f[1, --, m1, m2])
                   c1                      c1
```

...modification of the objects by Dilat symmetries, similar to the above ...

```
1(step 2) By the symmmetry Dilat{m1, m2}, the system proposed the
modification from the object:
                                        c2
                                t2 f[1, --, m1, m2]
  tf            c2                      c1
  -- - (1 - f[1, --, m1, m2] + -------------------)
  t1            c1                      t1
to:                                c2     m2
                           t2 f[1, --, 1, --]
tf            c2     m2            c1     m1
-- - (1 - f[1, --, 1, --] + ------------------)
t1            c1     m1              t1
1(step 3.1.1) By the symmmetry Perm{m1, c1, m2, c2}, the system
derived the constraint:
      c2     m2                m2     c2           c2     m2
 f[1, --, 1, --] != 0 && f[1, --, 1, --] == f[1, --, 1, --]
      c1     m1                m1     c1           c1     m1
```

---

[4]"The intrinsic angular momentum of a particle with zero rest-mass is parallel to its direction of motion, that is parallel to its velocity [Wigner 67]."

```
1(step 3.1.1) By the symmmetry Perm{m1, m2, c1, c2, t1, t2}, the
system derived the constraint:
     c2     m2              c1     m1              c2     m2
f[1, --, 1, --]!=0 && f[1, --, 1, --]== 1 - f[1, --, 1, --]
     c1     m1              c2     m2              c1     m1
1(step 3.1.2) By the constraint:
      c2     m2                m2     c2           c2     m2
 f[1, --, 1, --] != 0 && f[1, --, 1, --] == f[1, --, 1, --]
      c1     m1                m1     c1           c1     m1
the system derives the symmetry:
    m2  c2
Com{--, --}
    m1  c1
for the object:
      c2     m2
 f[1, --, 1, --]
      c1     m1
1(step 3.1.2) By the constraint:
     c2     m2              c1     m1              c2     m2
f[1, --, 1, --]!=0 && f[1, --, 1, --]== 1 - f[1, --, 1, --]
     c1     m1              c2     m2              c1     m1
the system derives the symmetry:
      m2  c2
ASCom{--, --}
      m1  c1
for the object:
      c2     m2
 f[1, --, 1, --]
      c1     m1
1(step 3.1.3) The system creates subgoal of checking the
derived symmetries:
    m2  c2             m2  c2
Com{--, --} and ASCom{--, --}
    m1  c1             m1  c1
for the internal object:
      c2     m2
 f[1, --, 1, --]
      c1     m1

1.1(step 1.1) The system identifies that the symmetry
      m2  c2           m2  c2
ASCom{--, --} and Com{--, --}is not satisfied by the internal object.
      m1  c1           m1  c1

1.1(step 2) By the symmmetry
    m2  c2
Com{--, --},
    m1  c1
the system proposed the modification from the object:
      c2     m2
 f[1, --, 1, --]
      c1     m1
to:         m2  c2
f[1, 1, com[--, --]]
            m1  c1
1.1(step 2) By the symmmetry
      m2  c2
ASCom{--, --},
      m1  c1
the system proposed the modification from the object:
            m2  c2
f[1, 1, com[--, --]]
            m1  c1
```

```
to:
    1
---------
    c2 m2
1 + -----
    c1 m1
1.2(step 1.1) The system identifies that the symmetry
    m2  c2            m2  c2
Com{--, --} and ASCom{--, --}
    m1  c1            m1  c1
is  satisfied by the internal object.
1.2(step 1.2) The system terminated the subgoal because there is no
symmetry that is not satisfied by the internal object.
2(step 1.1) The system identifies that all given symmetries are
satisfied by the object with new internal object.
2(step 1.2) The system terminated because there are
no other symmetries that are not satisfied by the object with new
internal object.
```

Specification by symmetry can apply not only to physics but to mathematics as well. As we have seen in section 3.2., the *Pythagorean theorem*, for example, can be specified by the geometrical symmetries of a right triangle. The application of this specification method is not limited to the equations of physical laws. It can also apply to the equation of a particular system if the system exhibits symmetry. As we have seen in section 3.3., it can apply to derive the equation of a pendulum that exhibits some obvious symmetry specific to the system structure.

In this section, we have discussed the specification and derivation of an object (formula). However, the symmetry-based reasoning can be applied to specify or derive the symmetry (transformations) as well. Einstein specified or derived the Lorentz transformation from the fact that it leaves an object (the speed of light) unchanged. He also noted that the transformation leaves unchanged the other object such as Maxwell-Herz equation [Einstein, 05]. Our symmetry-based reasoning also can check whether some other objects may have the symmetry.

5. Conclusions

In this study, we demonstrated how the *Symmetry as Guiding Constraints* principle can be used to automate reasoning in order to specify, derive, and generalize equations in physical laws. Nevertheless, symmetry itself still needs to be found.

There is not always one-directional mapping from symmetries (constraints) to objects (equations). Rather, in reality, the elaboration process may go back and forth with symmetries and objects constraining each other.

The symbolic equation serves as a powerful representation of knowledge that can be: checked if it has a certain symmetry, solved if variables and constants are specified, transformed into canonical form, and so forth. In order for these operations to give physical implications, the syntax as well as semantics of the equation must be included in the knowledge of the system. In this paper , we implemented the syntax on the formula-rewriting system *Mathematica* and the semantics on both the long-term and short-term memory of a production system.

Symmetry-based reasoning on the other representations such as diagrammatic representation should be further explored in the future.

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