Structural Analysis on Interval Matrices by Signed Digraph: Structural Perturbation Principle

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Key Words - Qualitative system theory; interval matrix; sign structure; qualitative reasoning; graph; matrix regularity; matrix rank; inverse stability; interval systems

Abstract

This note deals with linear interval systems whose parameters in system matrix are expressed by intervals. We formalize the system reasoning about structures of interval systems by the structural perturbation principle: the interval system would have the interval property when its underlying sign structure include the component that has the corresponding sign property and the norm of the rest of component (considered structural perturbation) is small enough. Several interval properties of interval matrix such as nonsingularity, rank and inverse stability will be discussed by applying the principle to the graphical conditions for the corresponding sign properties. The Klein model in economics is used as an illustrative example.

1 Introduction

Analysis and synthesis on systems with parameter uncertainty has been studied in system theory [1, 2, 3, 4, 5, 6, 7]. The level of uncertainty may be divided into four; numerical (most specific) specification, interval specification , sign specification, and zero/nonzero specification of parameters as shown in Figure 1. We call numerical, interval, sign and structured systems respectively, corresponding to these level of parameter specification. The motivations for the study of systems with uncertain parameters are as follows: (1) modeling errors are inevitable in modeling real systems which are often complex and large-scale, and (2) early phase of system design must allow undetermined values in parameters.

These results have several implications in *Qualitative Reasoning* about systems with qualitatively expressed interactions and states. In this note, we present a principle which can be used to reason about interval systems based on its sign structure. We briefly present some theoretical results to demonstrate how the principle can be applied to the static properties of the system matrix. This research aims to apply the results for one level to the other level in Figure 1. Particularly, this note focuses on the results for sign matrices that can be used to deduce the corresponding results for interval matrices by the structural perturbation principle (stated in the next section).

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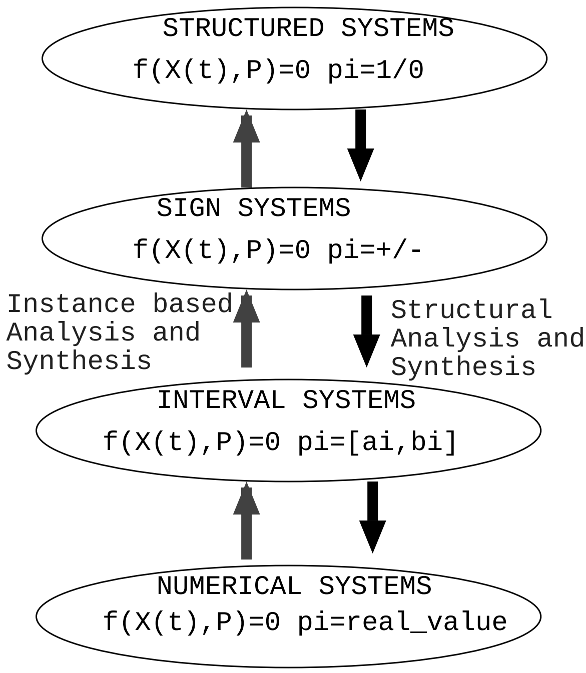

Figure 1: Levels of Systems based on Parameter Specification

Section 2 presents structural perturbation principle and some preliminary concepts required in the subsequent sections. Section 3 extends several properties such as nonsingularity, rank, and inverse stability to sign and interval matrices. Several results associating sign properties with corresponding interval properties are also presented. Section 4 demonstrates how the structural perturbation principle together with the results stated in section 3 can be applied to a specific system, taking the Klein model as an illustrative example. Section 5 discusses character and implication of this approach.

2 Preliminaries

Sign matrix A^S is a matrix whose elements are $+,-,0$. Interval matrix A^I is a matrix whose elements are specified by interval $[a,b]$ where a and b are two terminal values; $a \leq b$. An interval matrix can be considered as a set of matrices whose elements are in the intervals specified by the interval matrix. For simplicity, we focus on interval matrices whose intervals can be identified as $+,0$, or $-$. That is, we do not consider such interval as $[-2,5]$. Further, when the matrix is reducible [1], it can be reduced to irreducible components to each of which the analysis stated in this note can be applied independently. Thus, the matrix is assumed to be irreducible in the rest of this note.

Definition 2.1 (Signed Digraph for Matrices) *Signed directed graph (called signed digraph) of a matrix $A \in R^{n\times n}$ is a graph with n nodes and arcs directed from node i to node j with sign +(-) when $a_{ij} > 0 (< 0)$.*

Example 2.1

The signed digraph of the following interval matrix is shown in Figure 2.

$$\begin{pmatrix} [-2,-1] & [-2,-1] & [0,0] \\ [-2,-1] & [-4,-3] & [-5,-4] \\ [-3,-1] & [2,4] & [-2,-1] \end{pmatrix}$$

Definition 2.2 (Sign Nonsingularity and Interval Nonsingularity) *Sign matrix is called sign nonsingular if all the matrices having the sign structure is nonsingular. That is*

[1] Matrix A is called reducible if it can be decomposed into diagonal blocks by permutation matrix P:

$$PAP^T = \begin{pmatrix} \mathrm{A}_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}.$$

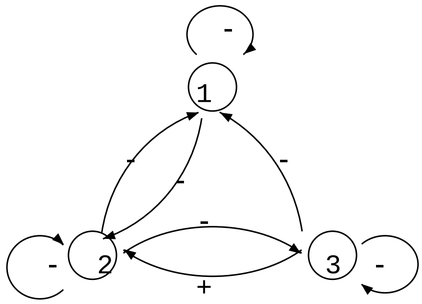

Figure 2: The Signed Digraph of the Interval Matrix

sign positive definite or sign negative definite[2]*. Interval matrix is called interval nonsingular if all the matrices whose elements are lying within the intervals are nonsingular.*

In the following, we present a principle that is used to reason about interval systems based on their sign structures. First, *Structural Inclusion Principle* may be stated as follows:

> For a system to have an interval property, the system must contain a subsystem that has the corresponding sign property.

This principle can apply to such properties as stability, observability, nonsingularity, inverse stability (that will be defined in the next section), solvability,*etc.* of system matrix. For example, application of the above structural inclusion principle to a system matrix for a property of nonsingularity results in the next statement; for an interval matrix to be interval nonsingular, its signed digraph must contain the arc-subgraph [3] that is sign nonsingular.

Due to the above structural inclusion principle, sign structures that generically [4] have the property may be decomposed into two subsystems: subsystem (arc-subgraph of the sign structure) that satisfies a sign property and a set of interactions (arcs) whose removal result in the system having the sign property. The set of interactions being removed is considered a structural perturbation imposed on the sign structure having the sign property. With this decomposition, *Structural Perturbation Principle* may be stated as:

> If a sign structure generically have the property but the interval system having the sign structure fails to have the corresponding interval property due to the

[2]Sign matrix A^S is sign positive (semi-)definite when all matrices $A \in A^S$ are positive (semi-)definite. Interval positive (semi-)definite can be defined in the same manner for interval matrices.

[3]Arc-subgraph is the graph obtained by deleting some arcs. Similarly, node-subgraph is the graph obtained by deleting some nodes and all the arcs associated with the nodes.

[4]An interval (sign, or structured) system is said to have a property generically when at least one instance of the interval (sign, or structured) system has the property.

structural perturbation, then the interval system would have the interval property if the structural perturbation is made small enough relative to the rest of system.

Again, applied to nonsingularity of matrix, the next statement follows: if the sign structure of an interval matrix is generically nonsingular, then the interval matrix can be made interval nonsingular by making the structural perturbation (the set of arcs whose removal leaves sign nonsingular graph) small.

The above principle apply to only well-posed property[5]. The principle applied to linear systems (i.e. matrix) can be stated as: a sign property of a sign matrix A^S may be formulated as $f(A^S) > \alpha$. Sign matrix A^S can be decomposed into structural perturbation $\Delta(A^S)$ and the rest of system $\tilde{A^S}$; $A^S = \Delta(A^S) + \tilde{A^S}$ where $f(\tilde{A^S}) > \alpha$ holds while $f(A^S) > \alpha$ does not hold due to the structural perturbation $\Delta(A^S)$. Then the corresponding interval property $f(A^I) > \alpha$ holds when the norm of $\Delta(A^I)$ is small enough to satisfy $min_{A \in A^I} f(A^I) > \alpha$. For nonsingularity of square sign matrices A^S, $f(A^S) > \alpha$ must be replaced with $|detA^S| > 0$.

This principle can be used not only in the analysis on whether the given interval system satisfy an interval property but also in the synthesis of parameters to satisfy the interval property based on its sign structure as will be discussed in section 5. Although we explained the principle applying from sign systems to interval systems, similar principle can be used from structured systems to sign systems and from interval systems to numerical systems (see Figure 1). We will be more specific how the principle can be used in the following sections by applying it to such properties as nonsingularity and inverse stability.

In the next section, we will present more informative measure than nonsingularity for sign and interval matrices. For singular matrices, rank would give information of how singular the matrix is. For nonsingular matrices, information of how many submatrices of order $n-1$ are nonsingular (and recursively for submatrices with the order $n-2, \cdots, 1$) would indicate how nonsingular the matrix is. The former singularity measure can be used to reason which arcs or nodes should be removed (in case of sign matrix) or made small (in case of interval matrix) to attain nonsingularity. The latter nonsingularity measure can be used to assess the properties stronger than nonsingularity such as inverse stability and solvability.

[5] That is, if the property holds with the parameter set P then it must hold in the sufficiently small neighborhood of the point P. Nonsingularity of a matrix is well-posed property, while singularity is not.

3 Properties of Static Systems: Rank, Nonsingularity, and Inverse Stability

In this section, we will explore sign properties and the corresponding interval properties. Conditions for these sign properties will be extended to those for corresponding interval properties by the structural perturbation principle. Sign matrices after put into the standard form with all the diagonal elements negative [6] is sign nonsingular if the corresponding signed digraph does not have any positive cycle [7] [1]. Thus, making all the positive cycles small[8] would make the interval matrix interval nonsingular by the structural perturbation principle.

Although it is necessary to cut all the positive cycles by deleting at least one arc from each positive arc to modify sign matrix which is not sign nonsingular but generically nonsingular (i.e. the signed digraph has arc-subgraph that is sign nonsingular) into sign nonsingular one, only some set of disjoint cycles stated in the following proposition 3.1 must be handled to control the interval nonsingularity.

Proposition 3.1 (Control of Interval Nonsingularity) *Interval matrix that is generically nonsingular can be made interval nonsingular by making the elements corresponding to the following arcs big enough; the arcs whose removal will cut all the cycles in the set of cycles* $\{C_{i1}, \cdots, C_{ip}\}$ *such that*

1. *these cycles are disjoint with each other,*
2. *the total sum of the length is* n, *and*
3. *the sign* $(-1)^p sgn(C_{i1} \cdot \cdots \cdot C_{ip})$ *is positive.*

One trivial set of cycles satisfying the conditions above is the set of all the negative loops (diagonal elements). Thus, making absolute value of all the diagonal elements big results in interval nonsingular matrices. This result also comes from nonsingularity condition of diagonal dominance.

Nonsingularity indicates whether or not information is preserved in the mapping by the matrix from a domain space to an image space. However, nonsingularity does not provide any further information about singular matrices. Rank of matrices is more informative

[6] For sign nonsingularity, the sign matrix must be put into the form with $a_{ii} < 0$ [1] by permissible qualitative operations[8]: exchange of i^{th} row (column) with j^{th} row (column) and multiplication of -1 to i^{th} row (column) with j^{th} row (column). This permissible qualitative operations are proved to preserve nonsingularity of matrix.

[7] Sign of a cycle is defined to be sign of the product of all the elements in the cycle.

[8] Cycle is said to be small(big) when the absolute value of the product of all the elements in the cycle is small but not zero(big). More specifically for interval matrices, when $b \leq cycle - product \leq a$, the cycle is said to be small(big) when $|a|, |b| \neq 0$ are small(big).

property than nonsingularity, since it can provide information about a kind of distance from nonsingularity.

Definition 3.1 (Sign Rank and Interval Rank) *Sign (interval) rank of a sign (interval) matrix is the order of its maximal submatrix that is sign(interval) nonsingular.*

Obviously, the interval rank of an interval matrix is greater than or equal to sign rank of the sing structure of the interval matrix. Sign rank can be obtained by signed digraph of the sign matrix as will be stated after proposition 3.2.

Although rank is more informative than nonsingularity, it cannot tell any further information for nonsingular (i.e. full rank) matrices. As it will be made clear through this section, information of nonsingularity of original matrix as well as submatrices of order $n-1$ is needed to characterize the concepts such as inverse stability discussed in this section.

Proposition 3.2 (Sign Nonsingularity of Submatrix) *Let N be a set of integers $\{1, 2, \cdots, n\}$ and I, J be a subset of N. If, for all the positive cycles and paths of opposite sign, there is at least one arc from the node $i \in I$ or to the node $j \in J$ that would cut the positive cycle or the paths of opposite sign, then the submatrix $A_{\bar{I}\bar{J}}$ is sign nonsingular where $\bar{I}$ and $\bar{J}$ are complement set of I and J. Here, $A_{\bar{I}\bar{J}}$ denotes the submatrix obtained by deleting all the i^{th} row in $i \in I$ and all the j^{th} column in $i \in J$ from A.*

The order of the maximal submatrix obtained by this proposition is sign rank.

In order to analyze the relation between a matrix and its submatrices, the relation between the graph corresponding to the original matrix and the subgraph corresponding to the submatrices may be used. A property of graph is said to be hereditary [10] if the property holds for its subgraph. For example, the property of having no cycle is hereditary. Graphical condition characterizing sign nonsingular matrix, that is having no positive cycle, is also hereditary (in the sense of node-subgraph). Thus, sign nonsingularity (for irreducible matrices) is hereditary as in the next theorem.

Theorem 3.3 (Sign Nonsingularity of Matrix and Submatrix) *If a sign matrix of order n is sign nonsingular then the submatrix A_{ij} of order $n-1$ are sign nonsingular for i, j such that $a_{ij} \neq 0$.*

Proof Nonsingularity of submatrix A_{ij} implies that (1) there is at least one non-zero path from node j to node i; $a_{ii_1} a_{i_1 i_2} \cdots a_{i_l j} \neq 0$ and (2) these non-zero paths must be of the same sign. The condition (1) is guaranteed by irreducibility of the original sign matrix of order n and (2) is guaranteed by sign nonsingularity of the original matrix. ∎

The above theorem 3.3 holds for only irreducible sign nonsingular matrices. If a reducible sign matrix of order n is sign nonsingular then its submatrix A_{ij} corresponding to $a_{ij} \neq 0$ of order $n-1$ are either sign nonsingular (i.e. sign rank is $n-1$) or structurally singular (i.e. term rank[9] $< n-1$). Although the number of sign nonsingular submatrices of order $n-1$ is at least the number of nonzero elements $a_{ij} \neq 0$ for sign nonsingular irreducible matrix of order n as known from the above theorem 3.3, that for sign nonsingular reducible matrix is at least n. This fact for reducible sign nonsingular matrix can be known from the cofactor expansion of A:

$$detA = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} A_{ij}.$$

An example that has exactly n sign nonsingular submatrices of order $n-1$ would be the sign matrix with all the diagonal elements negative and others zero.

This recursiveness of nonsingularity stated in theorem 3.3 does not hold for interval matrix with interval nonsingularity. That is, even if an interval matrix is interval nonsingular its submatrices may not be interval nonsingular as demonstrated in the next section. The following condition for interval nonsingularity also follows from the determinant expansion.

Proposition 3.4 (Interval Nonsingularity Condition by Submatrix) *If there exists i^{th} row (or j^{th} column) in the interval matrix such that $a_{ij} \neq 0$ always implies that A_{ij} is interval nonsingular and that $(-1)^{i+j} a_{ij} A_{ij}$ are of the same sign for j =1,$\cdots$, n(or i= 1,$\cdots$, n), then the original interval matrix is interval nonsingular.*

The following concept of inverse stability is related to sign(interval) nonsingularity of submatrices with order $n-1$.

Definition 3.2 (Inverse Stability) *A sign matrix A^S (interval matrix A^I) is called sign (interval) inverse stable when A^S (A^I) is sign (interval) nonsingular, and $|\{A^{-1}\}_{ij}| > 0$ for all $A \in A^S(A^I)$. That is, all the element of the inverse of the given matrix does not become zero. A sign matrix A^S (interval matrix A^I) is called sign (interval) inverse semi-stable when when A^S (A^I) is sign (interval) nonsingular, and $\{A^{-1}\}_{ij} \geq 0$ or $\{A^{-1}\}_{ij} \leq 0$ for all $A \in A^S(A^I)$. Here, $\{B\}_{ij}$ denotes ij-element of the matrix B.*

This inverse stability, originally defined on interval matrices [3], can be defined on sign matrices as above. Obviously, the inverse stability of sign (interval) matrices is equal to that all the submatrices of order $n-1$ as well as the original matrix of order n is sign (interval) nonsingular. The inverse semi-stability of sign (interval) matrix is equal to that the original matrix of order n are sign (interval) nonsingular and that all the submatrices of order $n-1$ are either sign(interval) positive semi-definite or sign(interval) negative semi-definite. Condition for inverse stability of interval matrices has been already obtained[4]. For sign matrices, the following result directly follows from theorem 3.3.

[9] Term rank for structured matrix where only zero/non-zero pattern is specified for each element is the maximal rank that the matrix satisfying the zero/non-zero pattern of the elements can attain. In other words, term rank fails to be full means that there is no non-zero term in its expansion of the determinant.

Table 1: Summary of Effects of Deleting or Weakening Perturbations

Deleting or Weakening Perturbations	Effect of Deleting or Weakening
Deletion of all the Positive Cycles[1]	Not Sign nonsingular $\rightarrow$ Sign nonsingular
Making Positive Cycles Small	Not interval nonsingular $\rightarrow$ Interval nonsingular
Making Arcs satisfying condition stated in Proposition 3.1 big	Not interval nonsingular $\rightarrow$ Interval nonsingular
Deletion of all the Paths of Opposite Sign of length smaller than k+1 (Proposition 3.2)	Sign rank $<$ k $\rightarrow$ Sign rank $\geq$ k
Addition of Paths of Opposite Sign satisfying the condition of Proposition 3.6	Sign inverse stable $\rightarrow$ Not sign inverse stable
Making Positive Cycles and Paths of Opposite Sign satisfying the condition of Proposition 3.6 small	Not interval inverse stable $\rightarrow$ Interval inverse stable
Deletion of Arcs resulting in Reducible Matrix (Corollary 3.5)	Sign inverse stable $\rightarrow$ Not sign inverse stable

Corollary 3.5 *A (reducible) sign matrix is inverse (semi-)stable if it is sign nonsingular and all the paths from node i to node j corresponding to $a_{ji} = 0$ are of the same sign.*

Result similar to corollary 3.5 does not hold for interval nonsingularity and inverse stability of interval matrices. Inverse stability condition for interval matrix consists of interval nonsingularity of original interval matrix and that of all the submatrices of order $n - 1$. Inverse stability of interval matrices plays an important role for specifying intervals of inverse of interval matrices as well as solving interval systems of linear equations [3, 4].

As known from the above corollary 3.5, irreducibility is necessary for inverse stability of sign matrices. In order to make the given sign matrix inverse stable, cutting the positive cycles and the paths of opposite sign from node i to node j such that $a_{ji} = 0$ does not always suffice (it does, however, for inverse semi-stability).

Proposition 3.6 *Irreducible sign matrix with positive cycles (hence not sing nonsingular) can be made inverse stable by deleting a set of arcs whose removal*

1. *leaves no positive cycle,*
2. *leaves no pair of paths of opposite sign from node i to node j such that $a_{ji} = 0$, and*
3. *preserves irreducibility.*

Table 2: Cycles and Critical Paths

	POSITIVE	NEGATIVE
Cycles of length greater than one	$a_{13}a_{34}a_{41}$, $a_{15}a_{54}a_{41}$, $a_{25}a_{54}a_{42}$	$a_{25}a_{53}a_{34}a_{42}$, $a_{15}a_{53}a_{34}a_{41}$
Paths from 1 to 2	$a_{15}a_{53}a_{34}a_{42}$	$a_{15}a_{54}a_{42}$,$a_{13}a_{34}a_{42}$
Paths from 1 to 3	$a_{15}a_{53}$	a_{13}
Paths from 2 to 1	$a_{25}a_{53}a_{34}a_{41}$	$a_{25}a_{54}a_{41}$
Paths from 2 to 3	$a_{25}a_{53}$	$a_{25}a_{54}a_{41}a_{13}$
Paths from 5 to 3	$a_{54}a_{41}a_{13}$	a_{53}
Paths from 5 to 4	a_{54}	$a_{53}a_{34}$

When applying the structural perturbation principle to the above proposition 3.6, irreducibility need not to be taken care of as noted before proposition 3.1. Interval inverse stability can be attained by making positive cycles and paths of opposite sign from node i to node j such that $a_{ji} = 0$ small, since irreducibility is always preserved by the operation of making the absolute values of matrix elements small.

In summary, sign properties discussed here can be identified by checking the sign of the determinant of original and submatrices, which can be done by checking signs of cycles and paths in the signed directed graph. We also discussed which interactions should be made small(or big) to attain corresponding interval properties by the structural perturbation principle. In order to know how small(or big) the interactions must be made, finding maximum and minimum value of the determinant of original and submatrices is needed. This can be done by using sign structure of interval matrices [9]. Table 1 summarizes the principle of perturbations presented in this section.

4 The Klein Model as an Illustrative Example

We use the Klein model in economics [11] as an illustrative example. Figure 3 shows the signed digraph of the irreducible component of the Klein model corresponding to the following interval matrix. Intervals are assigned arbitrary according to the sign structure of the Klein model.

$$\begin{pmatrix} [-3,-1] & [0,0] & [2,3] & [0,0] & [-5,-3] \\ [0,0] & [-7,-4] & [0,0] & [0,0] & [-1,-2] \\ [0,0] & [0,0] & [-8,-3] & [10,13] & [0,0] \\ [2,4] & [2,5] & [0,0] & [-7,-5] & [0,0] \\ [0,0] & [0,0] & [7,9] & [-5,-3] & [-4,-1] \end{pmatrix}$$

Table 2 lists all the cycles of length greater than one and the critical paths, that are paths from node i to node j for i,j such that $a_{ji} = 0$. In order to make the given sign matrix

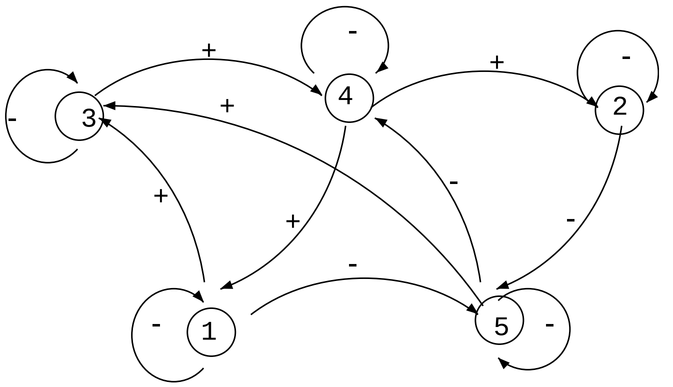

Figure 3: The Signed Digraph of the Irreducible Component of the Klein Model

(interval matrix) sign nonsingular (interval nonsingular), all the positive cycles must be cut (made small). In order to further make the sign nonsingular (interval nonsingular) matrix sign inverse stable (interval inverse stable), some arcs in the critical paths listed must be cut (made small) to make critical paths have the same sign, preserving irreducibility.

First, it is known that the sign structure of the interval matrix is not sign nonsingular due to the existence of positive cycles: $a_{25}a_{54}a_{42}, a_{13}a_{34}a_{41}, a_{15}a_{54}a_{41}$. Among the cut set of arcs whose removal will cut all these positive cycles, the sets $\{a_{54}, a_{41}\}$, $\{a_{54}, a_{34}\}$, $\{a_{54}, a_{13}\}$, $\{a_{41}, a_{42}\}$, $\{a_{24}, a_{41}\}$ are the minimal set. Thus, making either of these values small would make the interval matrix interval nonsingular by the structural perturbation principle. There is an arc contained only in the negative cycles; a_{53}. For example, the interval matrix obtained by replacing a_{53} of the above interval matrix with $[1, 9]$is not interval nonsingular. However, by proposition 3.1, we can make this interval matrix interval nonsingular by changing a_{53} from $[1, 9]$ to $[5, 9]$. Although this strategy can work to control interval nonsingularity, it does not work to control interval inverse stability. In fact, a_{53} does not appear in the submatrix A_{52} that is not sign nonsingular, hence may not be interval nonsingular.

By proposition 3.2, the following submatrices of order four turned out to be sign nonsingular; $A_{13}, A_{14}, A_{23}, A_{24}, A_{43}, A_{44}, A_{51}, A_{53}, A_{54}$. Thus, the sign rank of the given sign structure is known to be four.

Since A_{44} and A_{54} are sign nonsingular, the interval stability of the submatrix A_{34} implies interval nonsingularity of the given interval matrix by proposition 3.4. After A_{34} is known to be interval nonsingular either by the combinatorial search of min/max of determinant [9] or recursively analyzing the sign structure of the submatrix A_{34}, the given interval matrix is judged to be interval nonsingular. The status of entire submatrices of order four follows:

$$\begin{pmatrix} INS & INS & SNS & SNS & INS \\ NotINS & INS & SNS & SNS & INS \\ INS & NotINS & NotINS & INS & NotINS \\ NotINS & INS & SNS & SNS & INS \\ SNS & NotINS & SNS & SNS & NotINS \end{pmatrix}$$

where i^{th} row and j^{th} column indicates the status of the submatrix A_{ij}. NotINS, INS, and SNS, respectively indicate not interval nonsingular, interval nonsingular and sign nonsingular.

Among the cut set of positive cycles, the removal of only the set $\{a_{54}, a_{13}\}$ will leave the graph irreducible and no pair of paths of opposite sign from node i to node j such that $a_{ji} = 0$. Hence, by proposition 3.6, the removal of these two arcs will make the sign structure not only sign nonsingular but also sign inverse stable. By the structural perturbation principle, making these parameters small would result in the interval matrix interval nonsingular and interval inverse stable.

As known from the status of submatrices A_{ij} above, the given interval matrix is not interval inverse stable. However, changing intervals from $a_{13} = [2,3], a_{54} = [-5,-3]$ to $a_{13} = [1/10, 1/5], a_{54} = [-1/5, -1/10]$ would make it interval inverse stable. In this case, sign of all the elements of A^{-1} can be identified by the signs of paths from node i to node j and the sign of $detA$ as follows [10]:

$$\begin{pmatrix} - & + & + & + & + \\ + & - & + & + & + \\ - & - & - & - & + \\ - & - & + & - & + \\ - & - & - & - & - \end{pmatrix}$$

5 Discussions

Brouwer *et al.* [11] proposed two approaches for mixed sign analysis: (1) the top-down approach: starting from sign equations, then try to identify which and how many equations have to be estimated numerically, (2) the bottom-up approach: starting from fully estimated numerical equations, then try to identify which and how many equations can be specified in sign equations preserving sign properties.

We have demonstrated an alternative approach in the previous section for the same example (i.e. the Klein model) as [11]. That is, (3) the interval approach: starting from interval systems whose intervals are set rather wide, then try to make the intervals narrower.

[10] Sign of ij-element of A^{-1} is $sgn(detA)sgn((-1)^{i+j}A_{ji})$ where $sgn(detA)$ is $(-1)^{order-of-A}$ and $sgn((-1)^{i+j}A_{ji})$ is the sign of the path from node i to node j.

In general, the strong point of taking sign structure into account in the study of interval systems is twofold:

- Analysis: sign structure can help to investigate a certain property of a given interval system when the sign structure satisfies a certain condition, or the deviation from the structure is small enough. Investigation directly on interval systems requires expensive combinatorial search which can be avoided in the above cases.

- Synthesis: sign structure can help to reason which interactions are critical and which interactions should be made small (or big) for a certain property. This synthesis apply all the instances of interval systems having the same sign structure, while synthesis in numerical level must work directly on each instance of interval systems.

As we discussed elsewhere [9], this principle can be used to reduce the numerical computations on interval matrices. The structural perturbation principle discussed in this note can be applied to any sign properties whose graphical conditions are known. The principle would indicate which elements of interval systems are critical to attain the interval properties. However, this is just one of many ways to attain the interval properties. There could be other ways of attaining the interval properties other than making the critical elements small(or big).

6 Conclusion

We presented the structural perturbation principle that can be used to reason about interval properties of interval systems based on their sign structures. By the principle, it is known which interactions are critical or should be made small (or big) to attain certain interval properties. In this note, we focus on such properties of matrices as nonsingularity, rank and inverse stability. The same approach is currently undertaken to relate sign and interval properties of dynamic systems such as stability and observability.

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