Unconstrained Optimization Reformulations of Variational Inequality Problems

by

Nobuo Yamashita[†], Kouichi Taji[‡] and Masao Fukushima[‡]

Graduate School of Information Science,
Nara Institute of Science and Technology,
Ikoma, Nara 630-01, Japan.

August 18, 1995

Abstract. Recently, Peng considered a merit function for the variational inequality problem (VIP), which constitutes an unconstrained differentiable optimization reformulation of VIP. In this paper, we generalize the merit function proposed by Peng and study various properties of the generalized function. We give conditions under which any stationary point of the generalized merit function is a solution of VIP and conditions under which that function provides a global error bound for VIP. We also present a descent method for solving VIP based on the generalized merit function.

Key words: Variational inequality problems, unconstrained optimization reformulation, global error bound, descent method.

[†]The work of this author was supported by the Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists.

[‡] The work of these authors was supported in part by the Scientific Research Grant-in-Aid from the Ministry of Education, Science and Culture, Japan.

1. Introduction

The variational inequality problem (VIP) is to find a vector x in $S \subset R^n$ such that

$$\langle F(x), y - x \rangle \geq 0 \quad \text{for all } y \in S,$$

where F is a continuously differentiable function from R^n into itself, S is a nonempty closed convex set and $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n [6]. When S is the nonnegative orthant in R^n, this problem reduces to the nonlinear complementarity problem (NCP) [10].

A real-valued function defined on R^n or its subset is called a merit function for VIP, if the set of its global minimizers on a given set coincides with the set of solutions to the VIP. A merit function serves as a useful tool for developing algorithms to solve VIP and for analyzing their convergence properties. Thus the study of merit functions has become one of the important research subjects on VIP. (See [5] for a comprehensive survey of merit functions for VIP and NCP.) Many of the merit functions for VIP known to date stem from Auslender's gap function [2], which constitutes a constrained optimization problem equivalent to VIP. This function is, however, neither differentiable nor finite-valued in general. Later, Auchmuty [1] and Fukushima [4] independently and almost simultaneously presented differentiable merit functions for VIP. Specifically, Fukushima [4] considered the regularized gap function $f_\alpha : R^n \to R$ defined by

$$f_\alpha(x) = \max_{y \in S} \left\{ \langle F(x), x - y \rangle - \frac{\alpha}{2} \|x - y\|^2 \right\}, \tag{1}$$

where α is a positive parameter. (The Euclidean norm in (1) may be replaced by any norm induced by a positive definite symmetric matrix.) The function f_α also constitutes a constrained equivalent optimization reformulation of VIP in the sense that x solves the VIP if and only if x minimizes f_α on S and $f_\alpha(x) = 0$. The regularized gap function was further extended by Wu et al. [13]. However, whether or not there is an unconstrained differentiable optimization reformulation of VIP still remained as an open question. Recently, Yamashita and Fukushima [14] studied the Moreau-Yosida regularizations of some existing merit functions including the function f_α and showed that they constitute unconstrained differentiable optimization problems equivalent to VIP. Those functions have favorable theoretical properties but are not necessarily easy to evaluate in practice. More recently, Peng [11] considered the function $M_\alpha : R^n \to R$ defined by

$$M_\alpha(x) = f_{\frac{1}{\alpha}}(x) - f_\alpha(x), \tag{2}$$

where f_α is the regularized gap function defined by (1) and α is a constant such that $\alpha > 1$, and showed that the function M_α constitutes an unconstrained differentiable optimization reformulation of VIP. Peng [11] also showed that, when VIP is specialized to NCP, this function reduces to the implicit Lagrangian proposed by Mangasarian and Solodov [8]. Thus the function M_α defined by (2) can be considered as an extension of the implicit Lagrangian for NCP to VIP, and hence the function M_α inherits many properties from the implicit Lagrangian for NCP.

The purpose of this paper is to generalize the merit function M_α and study various properties of the generalized function. The generalization is two-fold. First, we use two arbitrary positive parameters α and β such that $\alpha < \beta$ to define the merit function, instead of $1/\alpha$ and α as in (2). Second, we adopt a generalized distance function, as in Wu et al. [13], in the definition of the regularized gap function. These extensions lend themselves to increasing the flexibility in designing the merit function in practice. We give conditions under which any stationary point of the generalized merit function is a solution of VIP and conditions under which that function provides a global error bound for VIP. Finally, we present a descent method for solving VIP based on the generalized merit function, which is an extension of the descent method for NCP proposed in [15].

2. Generalization of the regularized gap function

In this section, we review some basic properties of the generalization of the regularized gap function f_α, which was introduced by Wu et al. [13]. First we recall that the regularized gap function f_α defined by (1) has the following good properties [2, 4, 7, 12]: f_α is nonnegative on S; $f_\alpha(x) = 0$ and $x \in S$ if and only if x is a solution of VIP; f_α is continuously differentiable; and $\sqrt{f_\alpha}$ provides a global error bound on S, if F is strongly monotone.

Wu et al. [13] considered the following function $\hat{f}_\alpha : R^n \to R$, which generalized the function f_α:

$$\hat{f}_\alpha(x) = \max_{y \in S} \Psi_\alpha(x, y), \tag{3}$$

where

$$\Psi_\alpha(x, y) = \langle F(x), x - y \rangle - \alpha\phi(x, y) \tag{4}$$

with a positive constant α and a function $\phi : R^{2n} \to R$ satisfying the following conditions:

C.1 ϕ is continuously differentiable on R^{2n};

C.2 ϕ is nonnegative on R^{2n};

C.3 $\phi(x, \cdot\,)$ is strongly convex uniformly in x, i.e., there exists a constant $\lambda > 0$ such that, for any $x \in R^n$,

$$\phi(x, y_1) - \phi(x, y_2) \geq \langle \nabla_y \phi(x, y_2), y_1 - y_2 \rangle + \lambda \|y_1 - y_2\|^2 \quad \text{for all } y_1, y_2 \in R^n;$$

C.4 $\phi(x, y) = 0$ if and only if $x = y$.

Wu et al. [13] showed that the function $\hat{f}_\alpha$ enjoys most of the favorable properties of f_α: $\hat{f}_\alpha$ is nonnegative on S; $\hat{f}_\alpha(x) = 0$ and $x \in S$ if and only if x is a solution of VIP; $\hat{f}_\alpha$ is continuously differentiable and

$$\nabla \hat{f}_\alpha(x) = F(x) + \nabla F(x)(x - y_\alpha(x)) - \alpha \nabla_x \phi(x, y_\alpha(x)), \tag{5}$$

where $y_\alpha(x)$ denotes the unique minimizer of $-\Psi_\alpha(x, \cdot\,)$ on S; and $\sqrt{\hat{f}_\alpha}$ provides a global error bound on S, if F is strongly monotone and Lipschitz continuous and $\nabla \phi_y(x, \cdot\,)$ is Lipschitz continuous uniformly in x.

Examples of the functions satisfying C.1–C.4 include the following:

- $\phi_1(x, y) = \psi_1(x - y)$, where $\psi_1 : R^n \to R$ is nonnegative, continuously differentiable, strongly convex on R^n and $\psi_1(0) = 0$.
- $\phi_2(x, y) = \psi_2(y) - \psi_2(x) - \langle \nabla \psi_2(x), y - x \rangle$, where $\psi_2 : R^n \to R$ is twice continuously differentiable and strongly convex.
- $\phi_3(x, y) = \langle x - y, B(x)(x - y) \rangle$, where $B(x)$ is a continuously differentiable, symmetric and uniformly positive definite $n \times n$ matrix.

The following lemma will be used in the subsequent discussion.

Lemma 2.1 *Let the function ϕ satisfy C.1–C.4. Then, $\nabla_y \phi(x, y) = 0$ if and only if $x = y$.*

Proof. Suppose that $\nabla_y \phi(x,y) = 0$. Then, C.3 implies that y is the unique minimizer of $\phi(x, \cdot\,)$. Since $\phi(x, \cdot\,)$ always has the unique minimizer x by C.2 and C.4, it follows that $x = y$. Next, we suppose $x = y$. Then, by C.4, we have $\phi(x,y) = 0$. It follows from C.2 that y is a global minimizer of $\phi(x, \cdot\,)$. Hence, we have $\nabla_y \phi(x,y) = 0$. □

3. Unconstrained optimization reformulation

We consider the function $g_{\alpha\beta} : R^n \to R$ defined by

$$\begin{aligned} g_{\alpha\beta}(x) &= \hat{f}_\alpha(x) - \hat{f}_\beta(x) \\ &= \max_{y \in S} \Psi_\alpha(x,y) - \max_{y \in S} \Psi_\beta(x,y) \\ &= \langle F(x), y_\beta(x) - y_\alpha(x) \rangle + \beta\phi(x, y_\beta(x)) - \alpha\phi(x, y_\alpha(x)), \end{aligned} \tag{6}$$

where α and β are arbitrary parameters such that $0 < \alpha < \beta$, $\hat{f}_\alpha$ and Ψ_α are the functions defined by (3) and (4), respectively, and $y_\alpha(x)$ denotes the unique minimizer of $-\Psi_\alpha(x, \cdot)$ on S. In this section, we show various properties of $g_{\alpha\beta}$. In particular, the equivalence between VIP and the unconstrained minimization of $g_{\alpha\beta}$ and give conditions under which any stationary point of $g_{\alpha\beta}$ is a solution of VIP. We assume throughout that the function ϕ in (4) satisfies conditions C.1–C.4. When $\phi(x,y) = \frac{1}{2}\|x - y\|^2$, $\beta > 1$ and $\alpha = \frac{1}{\beta}$, $g_{\alpha\beta}$ reduces to the merit function M_β considered by Peng [11].

The differentiability of the function $g_{\alpha\beta}$ immediately follows from that of $\hat{f}_\alpha$ and $\hat{f}_\beta$.

Theorem 3.1 *Let the function ϕ satisfy C.1–C.4. Then $g_{\alpha\beta}$ is differentiable and*

$$\nabla g_{\alpha\beta}(x) = \nabla F(x)(y_\beta(x) - y_\alpha(x)) + \beta \nabla_x \phi(x, y_\beta(x)) - \alpha \nabla_x \phi(x, y_\alpha(x)).$$

Proof. The explicit representation of the gradient follows from (5). □

Next we state a key property for the analysis to follow.

Proposition 3.1 *Let the function ϕ satisfy C.3. Then we have*

$$(\beta - \alpha)\phi(x, y_\beta(x)) \le g_{\alpha\beta}(x) \le (\beta - \alpha)\phi(x, y_\alpha(x)) \qquad \text{for all } x \in R^n.$$

Proof. By definition, we have

$$\begin{aligned} g_{\alpha\beta}(x) &= \max_{y\in S} \Psi_\alpha(x,y) - \max_{y\in S} \Psi_\beta(x,y) \\ &= \langle F(x), x - y_\alpha(x)\rangle - \alpha\phi(x, y_\alpha(x)) - \langle F(x), x - y_\beta(x)\rangle + \beta\phi(x, y_\beta(x)) \\ &\geq \langle F(x), x - y_\beta(x)\rangle - \alpha\phi(x, y_\beta(x)) - \langle F(x), x - y_\beta(x)\rangle + \beta\phi(x, y_\beta(x)) \\ &\geq (\beta - \alpha)\phi(x, y_\beta(x)). \end{aligned}$$

We can show the other inequality in a similar way. □

The next important lemma is given in [13, Propostion 3.3], where the proof is attributed to Dafermos [3, Lemma 1]. The latter proof, however, assumes that function ϕ is quadratic. So, for completeness, we give a proof for the general case of ϕ.

Lemma 3.1 *Let the function ϕ satisfy C.1–C.4. Then, for any $\alpha > 0$, $x = y_\alpha(x)$ if and only if x is solution of VIP.*

Proof. Suppose that x is a solution of VIP. Then x satisfies the inequality

$$\langle F(x), y - x\rangle \geq 0 \quad \text{for all } y \in S. \tag{7}$$

Since $y_\alpha(x)$ minimizes $-\Psi_\alpha(x,\cdot)$ on S, the following first order optimality condition is satisfied:

$$\begin{aligned} &\langle -\nabla_y \Psi_\alpha(x, y_\alpha(x)), u - y_\alpha(x)\rangle \\ &\quad = \langle F(x) + \alpha\nabla_y\phi(x, y_\alpha(x)), u - y_\alpha(x)\rangle \geq 0 \quad \text{for all } u \in S. \end{aligned} \tag{8}$$

Since $x \in S$, (8) implies

$$\langle F(x) + \alpha\nabla_y\phi(x, y_\alpha(x)), x - y_\alpha(x)\rangle \geq 0.$$

On the other hand, since $y_\alpha(x) \in S$, it follows from (7) that

$$\langle F(x), y_\alpha(x) - x\rangle \geq 0.$$

Adding the last two inequalities, we have

$$\langle \alpha\nabla_y\phi(x, y_\alpha(x)), x - y_\alpha(x)\rangle \geq 0. \tag{9}$$

Furthermore, the strong convexity of $\phi(x, \cdot)$ implies

$$\begin{aligned} \langle \nabla_y \phi(x, y_\alpha(x)), x - y_\alpha(x) \rangle + \lambda \|x - y_\alpha(x)\|^2 &\leq \phi(x, x) - \phi(x, y_\alpha(x)) \\ &\leq 0, \end{aligned}$$

where the last inequality follows from C.2 and C.4. This inequality and (9) imply $\|x - y_\alpha(x)\| = 0$, that is, $x = y_\alpha(x)$.

Conversely, suppose that $x = y_\alpha(x)$. Then, we have from (8) and Lemma 2.1

$$\langle F(x), u - x \rangle \geq 0 \quad \text{for all } u \in S,$$

which implies that x is a solution of VIP. □

Now, we establish the equivalence between VIP and the unconstrained minimization of $g_{\alpha\beta}$.

Theorem 3.2 *Let the function ϕ satisfy C.1–C.4. Then $g_{\alpha\beta}$ is nonnegative on R^n. Moreover, $g_{\alpha\beta}(x) = 0$ if and only if x is a solution of VIP.*

Proof. By Proposition 3.1,

$$g_{\alpha\beta}(x) \geq (\beta - \alpha)\phi(x, y_\beta(x)) \quad \text{for all } x \in R^n. \tag{10}$$

It then follows from C.2 that $g_{\alpha\beta}$ is nonnegative on R^n.

Next, we show the last part of the theorem. First, suppose that $g_{\alpha\beta}(x) = 0$. Then, (10), C.2 and C.4 imply that $x = y_\beta(x)$. Hence, by Lemma 3.1, x is a solution of VIP. To prove the converse, suppose that x is solution of VIP. Then, by Lemma 3.1, we have $x = y_\alpha(x)$. It follows from C.4 that $\phi(x, y_\alpha(x)) = 0$. Then, since $g_{\alpha\beta}$ is nonnegative, we must have $g_{\alpha\beta}(x) = 0$ by Proposition 3.1. □

This theorem says that the unconstrained global minimization problem

$$\min_{x \in R^n} \quad g_{\alpha\beta}(x)$$

is equivalent to VIP, as long as the VIP has a solution. In particular, the unconstrained minimizer of $g_{\alpha\beta}$ with $g_{\alpha\beta}(x) = 0$ is shown to be contained in the set S automatically.

Note that the equivalence shown in Theorem 3.2 relies upon no specific assumption on F. For a general function F, however, $g_{\alpha\beta}$ may have stationary points which do not solve VIP. Because most

unconstrained minimization methods only find a stationary point, or a local minimum at best, of the objective function, it is important to know conditions under which any stationary point of $g_{\alpha\beta}$ is actually a solution of VIP. For this purpose, we need one more condition on ϕ.

C.5 For any $y \in R^n$, $\nabla_x \phi(x, y) = -\nabla_y \phi(x, y)$.

Note that the function ϕ_1 shown in Introduction satisfies C.1–C.5. Under these conditions on ϕ, we show the following lemma.

Lemma 3.2 *Let the function ϕ satisfy C.1–C.5. Then, for any $x \in R^n$, we have*

$$\langle y_\beta(x) - y_\alpha(x), \beta\nabla_x \phi(x, y_\beta(x)) - \alpha\nabla_x \phi(x, y_\alpha(x))\rangle \geq 0.$$

Proof. By C.5, we have

$$\begin{aligned} &\langle y_\beta(x) - y_\alpha(x), \beta\nabla_x \phi(x, y_\beta(x)) - \alpha\nabla_x \phi(x, y_\alpha(x))\rangle \\ &= \langle F(x) + \beta\nabla_y \phi(x, y_\beta(x)), y_\alpha(x) - y_\beta(x)\rangle \\ &\qquad + \langle F(x) + \alpha\nabla_y \phi(x, y_\alpha(x)), y_\beta(x) - y_\alpha(x)\rangle. \end{aligned} \tag{11}$$

Since $y_\beta(x)$ minimizes $-\Psi_\beta(x, \cdot)$ on S, the first order optimality condition (8) is satisfied. Hence, letting $u = y_\alpha(x)$ in (8), we see that the first term on the right-hand side of (11) is nonnegative. Similarly, the second term is also nonnegative. Consequently, we have the desired inequality. □

Now we are ready to answer the question of when a stationary point of $g_{\alpha\beta}$ is a solution of VIP.

Theorem 3.3 *Let the function ϕ satisfy C.1–C.5. Suppose that $\nabla g_{\alpha\beta}(x) = 0$. If $\nabla F(x)$ is positive definite, then x is a solution of VIP.*

Proof. By Theorem 3.1, we have

$$\begin{aligned} 0 &= \nabla g_{\alpha\beta}(x) \\ &= \nabla F(x)(y_\beta(x) - y_\alpha(x)) + \beta\nabla_x \phi(x, y_\beta(x)) - \alpha\nabla_x \phi(x, y_\alpha(x)). \end{aligned} \tag{12}$$

Multiplying (12) by $y_\beta(x) - y_\alpha(x)$, we have

$$\begin{aligned} 0 &= \langle y_\beta(x) - y_\alpha(x), \nabla F(x)(y_\beta(x) - y_\alpha(x))\rangle \\ &\qquad + \langle y_\beta(x) - y_\alpha(x), \beta\nabla_x \phi(x, y_\beta(x)) - \alpha\nabla_x \phi(x, y_\alpha(x))\rangle. \end{aligned} \tag{13}$$

By Lemma 3.2, the second term on the right-hand side of (13) is nonnegative. It follows from (13) and the positive definiteness of $\nabla F(x)$ that

$$y_\beta(x) - y_\alpha(x) = 0.$$

Hence, by (12), we have

$$\begin{aligned} 0 &= \beta\nabla_x\phi(x, y_\beta(x)) - \alpha\nabla_x\phi(x, y_\alpha(x)) \\ &= (\beta - \alpha)\nabla_x\phi(x, y_\alpha(x)). \end{aligned}$$

It then follows from C.5 that

$$\nabla_y\phi(x, y_\alpha(x)) = 0.$$

By Lemma 2.1, we have $x = y_\alpha(x)$. Consequently, by Lemma 3.1, x is a solution of VIP. □

The positive definiteness of $\nabla F(x)$ imposed in Theorem 3.3 cannot be replaced by the strict monotonicity assumption on F. To see this, let us borrow from [15] a simple example with $n = 1$ such that

$$F(x) = (x - 3)^3 + 1,$$

$S = [0, +\infty)$ and $\phi(x, y) = 1/2\|x - y\|^2$. Note that F is strictly monotone, but $\nabla F(x)$ fails to be positive definite at $x = 3$. This VIP has the unique solution $x = 2$. But, for $\alpha = 1/2$ and $\beta = 2$, $\nabla g_{\alpha\beta}(x) = 0$ is also satisfied by $x = 3$, which is not a solution of VIP.

4. Global error bound

In this section, we investigate conditions under which $g_{\alpha\beta}$ provides a global error bound for VIP. Throughout this section, we assume that the function ϕ satisfies the following condition.

C.6 $\nabla_y\phi(x, \cdot)$ is uniformly Lipschitz continuous, i.e., there exists a constant $M > 0$ such that for any $x \in R^n$,

$$\|\nabla_y\phi(x, y_1) - \nabla_y\phi(x, y_2)\| \le M\|y_1 - y_2\| \quad \text{for all } y_1, y_2 \in R^n.$$

We remark that the condition C.5 will not be needed in this section.

To establish the error bound results, we need the following two lemmas. The first lemma gives conditions under which the "natural" residual provides a global error bound on the whole space. To guarantee the latter property, in general, we require the strong monotonicity and Lipschitz continuity of F, as in [9, 13], which appear to be very restrictive in practice. In the same lemma, however, we show that the Lipschitz continuity of F can be removed when the set S is either compact or equal to the whole space R^n. The assumption that $S = R^n$ is very special, because then VIP reduces to the system of equations $F(x) = 0$. However, assuming the compactness of S may be justified when F is strongly monotone. In fact, when F is strongly monotone, VIP has a unique solution x^*, and hence, in practice, the problem may be equivalently restated by replacing the set S by its intersection with a sphere large enough to contain the solution x^*.

Lemma 4.1 *Let the function ϕ satisfy C.1–C.4 and C.6. Suppose that F is strongly monotone on R^n. Then, under either one of the following three assumptions, there exists a constant $c > 0$ such that*

$$\|x - x^*\| \leq c\|y_\beta(x) - x\| \qquad \text{for all } x \in R^n,$$

where x^* is the unique solution of VIP.

(a) *F is Lipschitz continuous on R^n.*

(b) *S is compact.*

(c) *$S = R^n$.*

Proof. (a) See the derivation of (27) in the proof of [13, Proposition 3.5]. Note that, although $x \in S$ is assumed in the latter proof, it remains true even if $x \notin S$.

(b) Since S is compact, there is a positive number $b > 0$ such that $S \subset \{x| \parallel x \parallel \leq b\}$. Let B denote the closed sphere centered at origin with radius $3b$, that is,

$$B = \{x| \parallel x \parallel \leq 3b\}.$$

Since F is continuously differentiable, F is Lipschitz continuous on B. Then, we can slightly modify the proof of part (a) to show that there is a positive number $c' > 0$ such that

$$\|x - x^*\| \quad \leq \quad c' \parallel x - y_\beta(x) \parallel \qquad \text{for all } x \in B. \tag{14}$$

Now, let x be an arbitrary point such that $x \notin B$, that is $\| x \| > 3b$. Then it follows from the triangle inequality that

$$\begin{aligned} \| x - x^* \| &\leq \| x - y_\beta(x) \| + \| y_\beta(x) \| + \| x^* \| \\ &\leq \| x - y_\beta(x) \| + 2b, \end{aligned} \tag{15}$$

where the last inequality follows from $x^* \in S$ and $y_\beta(x) \in S$, that is, $\| x^* \| \leq b$ and $\| y_\beta(x) \| \leq b$. Also, from the triangle inequality, we have

$$\begin{aligned} \| x - y_\beta(x) \| &\geq \| x \| - \| y_\beta(x) \| \\ &\geq 3b - b \\ &= 2b. \end{aligned} \tag{16}$$

Hence, it follows from (15) and (16) that

$$\|x - x^*\| \leq 2 \| x - y_\beta(x) \| \qquad \text{for all } x \notin B. \tag{17}$$

Setting $c = \max\{c', 2\}$, we have from (14) and (17)

$$\|x - x^*\| \leq c \| x - y_\beta(x) \| \qquad \text{for all } x \in R^n.$$

(c) Since $S = R^n$, we have $F(x^*) = 0$. Since $y_\beta(x)$ is the unique minimizer of $-\Psi_\beta(x, \cdot)$, we have

$$F(x) + \beta \nabla_y \phi(x, y_\beta(x)) = 0. \tag{18}$$

Moreover, we have

$$\begin{aligned} \mu \|x - x^*\|^2 &\leq \langle F(x) - F(x^*), x - x^* \rangle \\ &\leq \|\beta \nabla_y \phi(x, y_\beta(x))\| \|x - x^*\| \\ &= \beta \|\nabla_y \phi(x, y_\beta(x)) - \nabla_y \phi(x, x)\| \|x - x^*\| \\ &\leq \beta M \|x - y_\beta(x)\| \|x - x^*\|, \end{aligned}$$

where the first inequality follows from the strong monotonicity of F, the second inequality follows from $F(x^*) = 0$ and (18), the equality follows from Lemma 2.1 and the last inequality follows from C.6. Thus we obtain

$$\|x - x^*\| \leq \frac{\beta M}{\mu} \|x - y_\beta(x)\|.$$

Setting $c = \frac{\beta M}{\mu}$, we have the desired inequality. □

We remark that, in the case (c) $S = R^n$, the strong monotonicity assumption on F can be replaced by the weaker assumption that F is a uniform P-function on R^n, i.e., for some $\mu > 0$, there exists an index i such that

$$(x_i - y_i)(F_i(x) - F_i(y)) \geq \mu \|x - y\|^2 \quad \text{for all } x, y \in R^n.$$

Lemma 4.2 *Let the function ϕ satisfy C.1–C.4 and C.6. Then, we have*

$$\lambda \|x - y_\alpha(x)\|^2 \leq \phi(x, y_\alpha(x)) \leq \frac{M}{2} \|x - y_\alpha(x)\|^2 \quad \textit{for all } x \in R^n, \tag{19}$$

where $\lambda > 0$ and $M > 0$ are the parameters in C.3 and C.6, respectively. Moreover, we have

$$\lambda(\beta - \alpha)\|x - y_\beta(x)\|^2 \leq g_{\alpha\beta}(x) \leq \frac{M(\beta - \alpha)}{2} \|x - y_\alpha(x)\|^2 \quad \textit{for all } x \in R^n. \tag{20}$$

Proof. First, we show the left-side inequality of (19). Observe that

$$\begin{aligned} \phi(x, y_\alpha(x)) &= \phi(x, y_\alpha(x)) - \phi(x, x) \\ &\geq \langle \nabla_y \phi(x, x), y_\alpha(x) - x \rangle + \lambda \|x - y_\alpha(x)\|^2, \end{aligned} \tag{21}$$

where the equality follows from C.4 and the inequality follows from C.3. Since C.2 and C.4 imply

$$0 = \phi(x, x) = \min_{y \in R^n} \phi(x, y),$$

we have

$$\nabla_y \phi(x, x) = 0. \tag{22}$$

Hence, substituting (22) into (21), we obtain

$$\phi(x, y_\alpha(x)) \geq \lambda \|x - y_\alpha(x)\|^2.$$

Next, we show the right-side inequality of (19). By $\phi(x, x) = 0$ and (22), we have

$$\begin{aligned} \phi(x, y_\alpha(x)) &= \phi(x, y_\alpha(x)) - \phi(x, x) \\ &= \int_0^1 \langle \nabla_y \phi(x, x + t(y_\alpha(x) - x)), y_\alpha(x) - x \rangle dt \\ &= \int_0^1 \langle \nabla_y \phi(x, x + t(y_\alpha(x) - x)) - \nabla_y \phi(x, x), y_\alpha(x) - x \rangle dt \\ &\leq \int_0^1 M \|y_\alpha(x) - x\|^2 t dt \\ &= \frac{M}{2} \|y_\alpha(x) - x\|^2, \end{aligned}$$

where the last inequality follows from C.6.

Finally, Proposition 3.1, (20) and (19) yield

$$\lambda(\beta-\alpha)\|x-y_\beta(x)\|^2 \le g_{\alpha\beta}(x) \le \frac{M(\beta-\alpha)}{2}\|x-y_\alpha(x)\|^2 \quad \text{for all } x \in R^n. \qquad \square$$

From the above lemmas, we obtain the following global error bound result for VIP.

Theorem 4.1 *Let the function ϕ satisfy C.1–C.4 and C.6. Suppose that F is strongly monotone on R^n. Then, under either one of the following three assumptions, $\sqrt{g_{\alpha\beta}}$ provides a global error bound for VIP.*

(a) *F is Lipschitz continuous on R^n.*

(b) *S is compact.*

(c) *$S = R^n$.*

Proof. By Lemma 4.1 and (20) in Lemma 4.2, we have

$$\|x-x^*\| \le \frac{c}{\sqrt{\lambda(\beta-\alpha)}}\sqrt{g_{\alpha\beta}(x)} \quad \text{for all } x \in R^n,$$

where x^* is the unique solution of VIP. $\square$

Corollary 4.1 *Let the function ϕ satisfy C.1–C.4 and C.6. Suppose that F is strongly monotone on R^n. Then, under either one of the following three assumptions, we have*

$$\lim_{\|x\|\to\infty} g_{\alpha\beta}(x) = +\infty,$$

and hence the level sets of $g_{\alpha\beta}$ are compact.

(a) *F is Lipschitz continuous on R^n.*

(b) *S is compact.*

(c) *$S = R^n$.* $\square$

5. Descent method

In this section, we discuss an iterative method for solving VIP. Throughout this section, we assume that the function ϕ satisfies C.1–C.5.

Theorem 3.1 and Lemma 3.2 indicate that, when $\nabla F(x)$ is positive definite, the vector

$$d = y_\alpha(x) - y_\beta(x) \tag{23}$$

is a descent direction of $g_{\alpha\beta}$ at x, i.e., $d \neq 0$ and $\nabla g_{\alpha\beta}(x) \neq 0$ imply

$$\begin{aligned} \langle \nabla g_{\alpha\beta}(x), d \rangle &= \langle y_\beta(x) - y_\alpha(x), \nabla F(x)(y_\beta(x) - y_\alpha(x)) \rangle \\ &\quad + \langle y_\beta(x) - y_\alpha(x), -\alpha \nabla_x \phi(x, y_\alpha(x)) + \beta \nabla_x \phi(x, y_\beta(x)) \rangle \\ &< 0. \end{aligned}$$

So it may seem viable to consider a descent method which uses the vector d defined by (23) as a search direction. Such a method would generate a sequence converging to a point x such that $d = y_\alpha(x) - y_\beta(x) = 0$. However, only $y_\alpha(x) = y_\beta(x)$ does not ensure that x is a solution of VIP. (From the proof of Theorem 3.3, it is seen that if $y_\alpha(x) = y_\beta(x)$ and x is a stationary point of $g_{\alpha\beta}$, then x is a solution of VIP.) Therefore, as in the descent method proposed in [15], we adopt the vector

$$d = r(x) + \rho s(x)$$

as a search direction, where ρ is a sufficiently small positive constant and

$$\begin{aligned} r(x) &= y_\alpha(x) - y_\beta(x), \\ s(x) &= \alpha \nabla_x \phi(x, y_\alpha(x)) - \beta \nabla_x \phi(x, y_\beta(x)). \end{aligned}$$

Since the vector d does not involve the Jacobian matrix $\nabla F(x)$, d is usually much easier to evaluate than the gradient $\nabla g_{\alpha\beta}(x)$. The next lemma shows that, under the strongly monotonicity assumption on F, the vector d is actually a descent direction of the function $g_{\alpha\beta}$ at x. The proof of the lemma uses the same technique as that of [15, Lemma 3.1].

Lemma 5.1 *Let the function ϕ satisfy C.1–C.5. Suppose that F is strongly monotone on R^n. Let x^0 be an arbitrary point in R^n. Then there exists a positive constant σ such that, for each x in the*

level set $T(x^0) = \{x | g_{\alpha\beta}(x) \leq g_{\alpha\beta}(x^0)\}$, *the vector* $d = r(x) + \rho s(x)$ *satisfies*

$$\langle d, \nabla g_{\alpha\beta}(x) \rangle \leq -\sigma \left(\|r(x)\| + \rho \|s(x)\| \right)^2,$$

provided that ρ *is chosen sufficiently small.*

Proof. From Theorem 3.1, we have

$$\begin{aligned}
&\langle d, \nabla g_{\alpha\beta}(x) \rangle \\
&= \langle d, \nabla F(x)(y_\beta(x) - y_\alpha(x)) \rangle + \langle d, \beta \nabla_x \phi(x, y_\beta(x)) - \alpha \nabla_x \phi(x, y_\alpha(x)) \rangle \\
&= -\langle d, \nabla F(x) r(x) \rangle - \langle d, s(x) \rangle \\
&= -\langle r(x), \nabla F(x) r(x) \rangle - \rho \langle s(x), \nabla F(x) r(x) \rangle - \langle r(x), s(x) \rangle - \rho \|s(x)\|^2. \qquad (24)
\end{aligned}$$

Note that, by Corollary 4.1, the level set $T(x^0)$ is bounded and hence compact. Since $\nabla F(x)$ is continuous, there exists a constant $\tau > 0$ such that

$$\|\nabla F(x)\| \leq \tau \qquad \text{for all } x \in T(x^0).$$

So, we have

$$-\langle s(x), \nabla F(x) r(x) \rangle \leq \tau \|r(x)\| \|s(x)\| \quad \text{for all } x \in T(x^0). \qquad (25)$$

Since F is strongly monotone, there exists a constant $\mu > 0$ such that

$$\langle r(x), \nabla F(x) r(x) \rangle \geq \mu \|r(x)\|^2. \qquad (26)$$

By Lemma 3.2, we have

$$\langle r(x), s(x) \rangle \geq 0. \qquad (27)$$

Thus, it follows from (24)-(27) that

$$\langle d, \nabla g_{\alpha\beta}(x) \rangle \leq -\mu \|r(x)\|^2 + \tau\rho \|r(x)\| \|s(x)\| - \rho \|s(x)\|^2. \qquad (28)$$

Note that the right-hand side of (28) can be rewritten using an arbitrary real parameter η as

$$\begin{aligned}
&-\mu \|r(x)\|^2 + \tau\rho \|r(x)\| \|s(x)\| - \rho \|s(x)\|^2 \\
&= -\eta \left(\|r(x)\| + \rho \|s(x)\| \right)^2 - \frac{1}{2} \left(\sqrt{\mu} \|r(x)\| - \sqrt{\rho} \|s(x)\| \right)^2 \\
&\quad + \left(\eta - \frac{\mu}{2} \right) \|r(x)\|^2 + \left(\eta\rho - \frac{1}{2} \right) \rho \|s(x)\|^2 \\
&\quad + \sqrt{\rho} \left(2\eta\sqrt{\rho} + \tau\sqrt{\rho} - \sqrt{\mu} \right) \|r(x)\| \|s(x)\|. \qquad (29)
\end{aligned}$$

We can choose $\eta > 0$ and $\rho > 0$ small enough to satisfy

$$\eta - \frac{\mu}{2} \leq 0, \quad \eta\rho - \frac{1}{2} \leq 0, \quad 2\eta\sqrt{\rho} + \tau\sqrt{\rho} - \sqrt{\mu} \leq 0. \tag{30}$$

Then, it follows from (28), (29) and (30) that

$$\langle d, \nabla g_{\alpha\beta}(x) \rangle \leq -\eta \left(\|r(x)\| + \rho \|s(x)\| \right)^2,$$

and hence we obtain the desired inequality with $\sigma = \eta$. □

Now we state an algorithm that uses the search direction d.

Algorithm

Step 1: Select an initial point $x^0 \in R^n$ and parameters $\rho > 0$, $\gamma \in (0, 1)$ and $\kappa \in (0, 1)$. Set $k := 0$.

Step 2: If $g_{\alpha\beta}(x^k) = 0$, then stop. Otherwise, go to Step 3.

Step 3: Let $d^k := r(x^k) + \rho s(x^k)$.

Step 4: Let m be the smallest nonnegative integer such that

$$g_{\alpha\beta}(x^k + \gamma^m d^k) - g_{\alpha\beta}(x^k) \leq -\gamma^m \kappa \left(\|r(x^k)\| + \rho \|s(x^k)\| \right)^2$$

and set $x^{k+1} := x^k + t_k d^k$, where $t_k := \gamma^m$. Return to Step 2 with k replaced by $k + 1$.

Note that, by Lemma 5.1, the Armijo-type line search can always find a step size $t_k > 0$ for each k, provided that the parameter κ is chosen smaller than σ. We establish a convergence theorem for the algorithm.

Theorem 5.1 *Let the function ϕ satisfy C.1–C.5. Suppose that F is strongly monotone on R^n and that either F is Lipschitz continuous on R^n or S is compact or $S = R^n$. Suppose also that ρ is small enough to guarantee the descent condition given in Lemma 5.1 and κ is smaller than σ given in Lemma 5.1. Then the sequence $\{x^k\}$ generated by the algorithm converges to the unique solution of VIP.*

Proof. By the descent property of the algorithm, the generated sequence $\{x^k\}$ is contained in the level set $T(x^0) = \{x | g_{\alpha\beta}(x) \leq g_{\alpha\beta}(x^0)\}$. By Corollary 4.1, the level set $T(x^0)$ is compact, which

implies that there exists at least one accumulation point of $\{x^k\}$. Since the sequence $\{g_{\alpha\beta}(x^k)\}$ is nonnegative and decreases monotonically, it converges to some $g^*_{\alpha\beta} \geq 0$. We assume $g^*_{\alpha\beta} > 0$. Let x^* be any accumulation point of $\{x^k\}$ and $\{x^k\}_{k\in K}$ be a subsequence converging to x^*. Note that, since $y_\alpha(\cdot)$ and $y_\beta(\cdot)$ are continuous, the boundedness of $\{x^k\}$ implies the boundedness of $\{r(x^k)\}$ and $\{s(x^k)\}$, which in turn implies the boundedness of $\{d^k\}$. Taking a further subsequence if necessary, we can assume without loss of generality that $d^k \to d^*$. Note that $d^* = r(x^*) + \rho s(x^*)$, since r and s are continuous. By the line search rule of the algorithm, we have

$$g_{\alpha\beta}(x^k + t_k d^k) - g_{\alpha\beta}(x^k) \leq -t_k \kappa \left(\|r(x^k)\| + \rho \|s(x^k)\| \right)^2,$$

which implies that $\{t_k(\|r(x^k)\| + \rho\|s(x^k)\|)^2\}$ converges to 0. If $\{t_k\}$ is bounded away from 0, then we have $|r(x^k)\| + \rho\|s(x^k)\| \to 0$, i.e., $|r(x^*)\| + \rho\|s(x^*)\| = 0$. Now suppose that there exists a subsequence such that $t_k \to 0$. The line search rule indicates

$$\frac{g_{\alpha\beta}(x^k + t'_k d^k) - g_{\alpha\beta}(x^k)}{t'_k} > -\kappa \left(\|r(x^k)\| + \rho \|s(x^k)\| \right)^2, \tag{31}$$

where $t'_k = t_k/\gamma$. Since $t'_k \to 0$, taking the limit of both sides of (31) yields

$$\langle \nabla g_{\alpha\beta}(x^*), d^* \rangle \geq -\kappa \left(\|r(x^*)\| + \rho \|s(x^*)\| \right)^2. \tag{32}$$

Since $d^* = r(x^*) + \rho s(x^*)$ as noted above, Lemma 5.1 and (32) imply

$$-\kappa \left(\|r(x^*)\| + \rho \|s(x^*)\| \right)^2 \leq -\sigma \left(\|r(x^*)\| + \rho \|s(x^*)\| \right)^2. \tag{33}$$

Since $\kappa < \sigma$ by assumption, (33) implies $\|r(x^*)\| + \rho\|s(x^*)\| = 0$. Consequently, we have $r(x^*) = 0$ and $s(x^*) = 0$. Since $r(x^*) = y_\alpha(x^*) - y_\beta(x^*)$ and $s(x^*) = \alpha \nabla_x \phi(x, y_\alpha(x^*)) - \beta \nabla_x \phi(x, y_\beta(x^*))$, Theorem 3.1 indicates that $\nabla g_{\alpha\beta}(x^*) = 0$. Thus it follows from Theorem 3.3 that the accumulation point x^* of $\{x^k\}$ solves VIP, and hence, by Theorem 3.2, $g_{\alpha\beta}(x^*) = 0$. This is contradictory to our hypothesis $g_{\alpha\beta}(x^*) = g^*_{\alpha\beta} > 0$. Therefore, we must have $g_{\alpha\beta}(x^k) \to 0$, which implies that any accumulation point x^* of $\{x^k\}$ satisfies $g_{\alpha\beta}(x^*) = 0$, and hence solves VIP. Since the strong monotonicity of F guarantees that the solution of VIP is unique, we can conclude that the entire sequence $\{x^k\}$ converges to the solution of VIP. □

6. Concluding remarks

In this paper, we have generalized Peng's merit function [11] for VIP and shown that the generalized function enjoys various important properties. We have presented a descent method based on this merit function and established a global convergence theorem. The results obtained in this paper suggest that this merit function may serve as a powerful tool in the study of VIP.

In Theorem 4.1, we have assumed the Lipschitz continuity of F as well as the strong monotonicity in order to show that $\sqrt{g_{\alpha\beta}}$ provides a global error bound, when no special assumption on S is made. However, we have also shown that, in the special case $S = R^n$ or S is compact, $\sqrt{g_{\alpha\beta}}$ provides a global error bound without the Lipschitz condition on F. So, the following question is raised. Whether or not does $\sqrt{g_{\alpha\beta}}$ provide a global error bound under the strong monotonicity of F only, even if S is a general convex set?

References

[1] Auchmuty, G., Variational principles for variational inequalities, *Numerical Functional Analysis and Optimization*, Vol. 10, pp. 863–874, 1989.

[2] Auslender, A., *Optimisation: Méthodes Numériques*, Masson, Paris, 1976.

[3] Dafermos, S., Traffic equilibrium and variational inequalities, *Transportation Science*, Vol. 13, pp. 42–54, 1980.

[4] Fukushima, M., Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems, *Mathematical Programming*, Vol. 53, pp. 99–110, 1992.

[5] Fukushima, M., Merit functions for variational inequality and complementarity problems, TR-IS-95022, Nara Institute of Science and Technology, Nara, Japan (1995).

[6] Harker, P.T. and Pang, J.-S., Finite-dimensional variational inequality and nonlinear complementarity problem: A survey of theory, algorithms and applications, *Mathematical Programming*, Vol.48, pp.161–220, 1990.

[7] Larsson, T. and Patriksson, M., A class of gap functions for variational inequalities, *Mathematical Programming*, Vol. 64, pp. 53–79, 1994.

[8] Mangasarian, O.L. and Solodov, M.V., Nonlinear complementarity as unconstrained and constrained minimization, *Mathematical Programming*, Vol. 62, pp. 277–297, 1993.

[9] Pang, J.-S., A posteriori error bounds for the linearly–constrained variational inequality problem, *Mathematics of Operations Research*, Vol. 12, pp. 474–484, 1987.

[10] Pang, J.-S., Complementarity Problems, *Handbook of Global Optimization*, Edited by R. Horst and P. Pardalos, Kluwer Academic Publishers, Norwell, MA, pp. 271–338, 1994.

[11] Peng, J.M., Equivalence of variational inequality problems to unconstrained optimization, Technical Report, State Key Laboratory of Scientific and Engineering Computing, Academia Sinica, Beijing, China, 1995.

[12] Taji, K., Fukushima, M. and Ibaraki, T., A globally convergent Newton method for solving strongly monotone variational inequalities, *Mathematical Programming*, Vol. 58, pp. 369–383, 1993.

[13] Wu, J.H., Florian, M. and Marcotte, P., A general descent framework for the monotone variational inequality problem, *Mathematical Programming*, Vol. 61, pp. 281–300, 1993.

[14] Yamashita, N. and Fukushima, M., Equivalent unconstrained minimization and global error bounds for variational inequality problems, TR-IS-94034, Nara Institute of Science and Technology, Nara, Japan (1994).

[15] Yamashita, N. and Fukushima, M., On stationary points of the implicit Lagrangian for nonlinear complementarity problems, *Journal of Optimization Theory and Applications*, Vol. 84, pp. 653–663, 1995.