The weight distributions of cosets of the second-order Reed-Muller code of length 128 in the third-order Reed-Muller code of length 128

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SUMMARY

In this paper, cosets of the second order Reed-Muller code of length 2^m, denoted $\mathrm{RM}_{m,2}$, in the third order Reed-Muller code of the same length, denoted $\mathrm{RM}_{m,3}$, are studied. The set of cosets, $\mathrm{RM}_{m,3}/\mathrm{RM}_{m,2}$ is partitioned into blocks. Two cosets are in the same block, if and only if there is a transformation in the general linear group by which one coset is transformed into the other. Two cosets in the same block have the same weight distribution. For the code length less than or equal to 128, the representative coset leader of each block is presented and the weight distribution of cosets in the block is computed. By using these results, the extended code of a cyclic code of length 128 between $\mathrm{RM}_{7,2}$ and $\mathrm{RM}_{7,2}$ can be decomposed into a set of cosets in $\mathrm{RM}_{7,3}/\mathrm{RM}_{7,2}$, and its weight distribution can be derived. Several cyclic codes of length 127 are shown to be equivalent and some new linear unequal error protection codes are found.

1. Introduction

It is important to know the weight distribution of (N, K) linear code C for its error performance analysis. A formula for the weight distribution of a code is only known for limited classes of codes, say, MDS codes and single, double and triple error-correcting binary primitive BCH codes. For other codes, theoretically we can compute the weight distribution of C by examining all its 2^K codewords or all the 2^{N-K} codewords of its dual code. If we do this by generating all linear combinations of row vectors from a generator matrix (or a parity-check matrix), the time complexity is $O(N2^K)$ (or $O(N2^{N-K})$).

In [2], we have proposed a method for computing the number of codewords of weight less than or equal to a given integer in a block code by using its trellis diagram. In [5], we have shown a method for computing the weight distribution of a code by using the weight distributions of the cosets with respect to its subcode. In [3, 4], we have also presented a recursive procedure based on the coset weight distributions with respect to a subcode of a projection of the code. By using these methods we have computed the weight distributions of (128, 43) and (128, 50) extended binary primitive BCH codes and the dual codes of (128, 78) and (128, 85) extended binary primitive BCH codes.

In this paper, we consider cosets of the second order Reed-Muller code of length 2^m, denoted $\mathrm{RM}_{m,3}$, which are included in the third order Reed-Muller code of the same length, denoted $\mathrm{RM}_{m,3}$. We partition the set of cosets into blocks. Two cosets are in the same block, if and only if there is a transformation in the general linear group [7] by which one coset is transformed into the other. Two cosets in the same block have the same weight distribution. For the code length less than or equal to 128, the representative coset leader of each block and the weight distribution of coset in the block are presented.

The results obtained here can be used for decomposing the extended code of a cyclic code of 127 between $\mathrm{RM}_{7,2}$ and $\mathrm{RM}_{7,3}$ into a set of cosets in $\mathrm{RM}_{7,3}/\mathrm{RM}_{7,2}$ and deriving its weight distribution. Several cyclic codes are shown to be equivalent. Some new linear unequal error protection (LUEP) codes [8] are also constructed.

2. Boolean polynomials of degree 3

For $0 \leq r \leq m$, let $P_{m,r}$ be the set of polynomials of degree r or less over GF(2) with variables $x_1, x_2, \ldots, x_m$. For $f = a_0 + \sum_{j=1}^{m} a_j x_j \in P_{m,1}$, f is expressed as

$$f = a_0 + \mathbf{a}\mathbf{x}^T,$$

where $\mathbf{a} = (a_1, a_2, \ldots, a_m)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m)$. For $f_i = a_{i0} + \mathbf{a}_i\mathbf{x}^T \in P_{m,1}$ with $1 \leq i \leq h$, let $l(f_1, f_2, \cdots, f_h)$ denote the number of independent vectors among $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_h$. If $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h$ are linearly independent, then $f_i = a_{i0} + \mathbf{a}_i\mathbf{x}^T$ with $1 \leq i \leq h$ are said to be linearly independent. Let $M_{m,r}$ be the set of all products of r linearly independent polynomials of degree one and let $\bar{M}_{m,r}$ be the set of all linear sums of polynomials in $M_{m,r}$ including $\mathbf{0}$. For a sum F in $\bar{M}_{m,r}$, let $\nu(F)$ denote the number of those products in $M_{m,r}$ which appear in F, and let $F(\mathbf{x})$ with $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ denote the polynomial in $P_{m,r}$ represented by F. For $F(x_1, x_2, \ldots, x_m)$ and an integer i with $1 \leq i \leq m$, let $F_{x_i=0}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$ (or simply $F_{x_i=0}$) denote $F(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_m)$. For an $m \times m$ matrix A over GF(2) whose (i, j) element is a_{ij}, $F(A\mathbf{x}^T)$ denotes the polynomial obtained from $F(\mathbf{x})$ by substituting $\sum_{j=1}^{m} a_{ij}x_j$ for each occurrence of x_i with $1 \leq i \leq m$. For F and G in $\bar{M}_{m,r}$ with $r \geq 1$, F and G are said to be equivalent (we write $F \approx G$), if and only if there is an invertible $m \times m$ matrix A over GF(2) such that

$$F(A\mathbf{x}^T) + G(\mathbf{x}) \in P_{m,r-1}.$$

If $F(A\mathbf{x}^T) + F(\mathbf{x}) \in P_{m,r-1}$, then A is said to be invariant linear transformation of F. For $F \in \bar{M}_{m,r}$, let $e(F)$ denote the equivalent class containing F and we choose a sum with the smallest number of products in the equivalent class as its representative.

For $f = \Pi_{i=1}^{r}(a_{i0} + \mathbf{a}_i\mathbf{x}^T) \in M_{m,r}$, let $L(f)$ be defined as the linear subspace spanned by $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r\}$ and for a linear subspace L of m-dimensional linear space V_m, $\#L$ denotes the dimension of L.

Lemma 1: Let f and g be in $M_{m,r}$.

(1) If $L(f) = L(g)$, then

$$f \approx g.$$

(2) If $\#(L(f) \cap L(g)) = r - 1$, then there is $h \in M_{m,r}$ such that

$$f + g \approx h.$$

$\Delta\Delta$

Hereafter, for $F = \sum_{i=1}^{h} f_i \in \bar{M}_{m,r}$, assume that

$$f_i \in M_{m,r}, \qquad \text{for } 1 \leq i \leq h, \tag{2.1}$$

$$\#(L(f_i) \cap L(f_j)) < r-1, \qquad \text{for } 1 \leq i < j \leq h. \tag{2.2}$$

Lemma 2: For $F = \sum_{i=1}^{h} f_i \in \bar{M}_{m,r}$, suppose that $\#(\cap_{i=1}^{h} L(f_i)) = r-2$. Then $F(\mathbf{x}) \approx 0$ or

$$F(\mathbf{x}) \approx x_1 x_2 \cdots x_{r-2}(x_{r-1}x_r + x_{r+1}x_{r+2} + \cdots + x_{r+2j-1}x_{r+2j}), \tag{2.3}$$

where $0 \leq j \leq h-1$ and $r+2j \leq m$. $\Delta\Delta$

Lemma 3: For

$$F = f_1 f_2 + f_3 f_4 + \cdots + f_{2h-1} f_{2h} \in \bar{M}_{m,2},$$

where $f_i = a_{i0} + \sum_{j=1}^{m} a_{ij}x_j$ for $1 \leq i \leq 2h$, suppose that $\#(\cap_{i=1}^{h} L(f_{2i-1}f_{2i})) = 0$. Then, $2h \leq m$ and there are a permutation p on $\{1, 2, \cdots, m\}$ and $g_i = \sum_{j=1}^{m} b_{ij}x_{p(j)}$ with $1 \leq i \leq 2h$ such that

$$\begin{cases} b_{ii} = 1, & \text{for } 1 \leq i \leq 2h, \\ b_{ij} = 0, & \text{for } 1 \leq i \leq 2h \text{ and } 1 \leq j < i, \\ b_{2i-1,2i} = 0, & \text{for } 1 \leq i \leq h, \end{cases} \tag{2.4}$$

and

$$F + g_1 g_2 + g_3 g_4 + \cdots + g_{2h-1} g_{2h} \in \bar{M}_{m,1}. \tag{2.5}$$

(Proof) Without loss of generality, assume that $a_{11} = 1$. By replacing f_{2i+2} with $f_{2i+1} + f_{2i+2}$, if necessary, we can assume that either $a_{2i+1,1} = a_{2i+2,1} = 0$ or $a_{2i+1,1} = 1$ and $a_{2i+2,1} = 0$, for $0 \leq i < h$. For every $i > 1$ such that $a_{i1} = 1$, replace f_i and f_2 as follows:

$$\begin{aligned} f_i &\leftarrow f_i + f_1, \\ f_2 &\leftarrow f_2 + \sum_{j \in \{i : a_{i1} = 1, 1 < i < 2h\}} f_{j+1}. \end{aligned}$$

Then, we can assume that

$$a_{i1} = 0, \qquad \text{for } 1 < i \leq 2h. \tag{2.6}$$

Without loss of generality, assume that $a_{22} = 1$. Again, we can assume that $a_{12} = 0$ and either $a_{2i+1,2} = a_{2i+2,2} = 0$ or $a_{2i+1,2} = 0$ and $a_{2i+2,2} = 1$ for $1 \leq i < h$ together with (2.6). For every $i > 2$ such that $a_{i2} = 1$, replace f_i and f_1 as follows:

$$\begin{aligned} f_i &\leftarrow f_i + f_2, \\ f_1 &\leftarrow f_1 + \sum_{j \in \{i : a_{i2} = 1, 2 < i < 2h\}} f_{j+1}. \end{aligned}$$

Note that condition (2.6) is kept. We can assume that

$$\begin{cases} a_{11} = 1, \\ a_{i1} = 0, \quad \text{for } 2 \leq i \leq 2h, \\ a_{22} = 1, \\ a_{i2} = 0, \quad \text{for } i = 1 \text{ and } 3 \leq i \leq 2h, \end{cases}$$

Repeat the same argument to $f_3 f_4 + \cdots f_{2h-1} f_{2h}$. $\Delta\Delta$

We have the following corollaries to Lemma 3.

Corollary 1: If $m = 2h$, then there is j such that $1 \leq j \leq m$, $g_1 = x_j$ and $b_{ij} = 0$ for $2 \leq i \leq 2h$.

(Proof) Lemma 3 implies that there is j such that $g_{2h} = x_j$. By changing the suffices of g_i's, $g_1 = x_j$. By applying Lemma 3 again, we have the corollary.

Corollary 2: Let $m = 6$ and $h = 2$. Then there are $h_i^{(1)} = \sum_{j=1}^{3} c_{ij} x_j$ and $h_i^{(2)} = \sum_{j=4}^{6} c_{ij} x_j$ with $2 \leq i \leq 4$ such that

$$(f_1 f_2 + f_3 f_4) + h_1^{(1)}(h_2^{(1)} + h_2^{(2)}) + (h_3^{(1)} + h_3^{(2)})(h_4^{(1)} + h_4^{(2)}) \in \bar{M}_{6,1}. \tag{2.7}$$

(Proof) For $1 \leq i \leq 4$, let $f_i^{(1)}$ and $f_i^{(2)}$ be defined as

$$f_i^{(1)} = \sum_{j=1}^{3} a_{ij} x_j \quad \text{and} \quad f_i^{(2)} = \sum_{j=4}^{6} a_{ij} x_j. \tag{2.8}$$

Then, one of $f_i^{(2)}$'s, say $f_1^{(2)}$, is expressed as

$$f_1^{(2)} = \sum_{i=2}^{4} c_i f_i^{(2)}. \tag{2.9}$$

Since $(f_1f_2+f_3f_4)+[(f_1+\sum_{i=2}^{4}c_if_i)f_2+(f_3+c_4f_2)(f_4+c_3f_2)] \in \bar{M}_{6,1}$, define $h_i^{(l)}$ with $1 \leq l \leq 2$ and $1 \leq i \leq 4$ as follows:

$$h_1^{(1)} \triangleq f_1^{(1)} + \sum_{i=2}^{4} c_i f_i^{(1)}, \quad h_1^{(2)} = 0, \tag{2.10}$$

$$h_2^{(l)} \triangleq f_2^{(l)}, \tag{2.11}$$

$$h_3^{(l)} \triangleq f_3^{(l)} + c_4 f_2^{(l)}, \tag{2.12}$$

$$h_4^{(l)} \triangleq f_4^{(l)} + c_3 f_2^{(l)}. \tag{2.13}$$

$\Delta\Delta$

Lemma 4: For $F \in \bar{M}_{m,3}$, $F(\mathbf{x})$ can be expressed as

$$F(x_1, x_2, \ldots, x_m) \approx G(x_1, x_2, \ldots, x_{m-1}) + x_m(g_1g_2 + g_3g_4 + \cdots + g_{2h-1}g_{2h}), \tag{2.14}$$

where $G \in \bar{M}_{m-1,3}$ and $g_1, g_2, \ldots, g_{2h}$ with $2h < m$ are linearly independent. $\Delta\Delta$

For $F \triangleq \sum_{i=1}^{h} f_i \in \bar{M}_{m,3}$, let f_i be expressed as $f_{i1}f_{i2}f_{i3}$ where $f_{ij} \in M_{m,1}$ for $1 \leq j \leq 3$, and let $m(F)$ denote the number of linearly independent polynomials in $\{f_{ij} : 1 \leq i \leq h \text{ and } 1 \leq j \leq 3\}$.

For

$$F(x_1, x_2, \ldots, x_m) = G(x_1, x_2, \ldots, x_{m-1}) + x_m(g_1g_2 + g_3g_4 + \cdots + g_{2h-1}g_{2h}), \tag{2.15}$$

Let $\nu_1(F)$ and $\nu_2(F)$ denote $\nu(G)$ and h in (2.15), respectively. As a representative E of $e(F)$, we choose E such that

(1) $m(E)$ is smallest,

(2) $\nu(E)$ is smallest under condition (1),

and (3) $\nu_2(E)$ is smallest under conditions (1) and (2).

Let $\tilde{m}(F)$, $\tilde{\nu}(F)$ and $\tilde{\nu}_2(F)$ denote $m(E)$, $\nu(E)$ and $\nu_2(E)$, respectively, where E is a representative of $e(F)$. The representative E is labeled $[m.\nu(E).\nu_2(E)\text{-}i]$, where i is the serial number for the same m, $\nu(E)$ and $\nu_2(E)$. For some cases, $\nu_2(E)$ (and/or i) is omitted.

Example 1: Let $F \in \bar{M}_{m,3}$ with $m \geq 3$.

1) $m(F) \leq 4$. Then, $\tilde{\nu}(F) = 1$ and $\tilde{m}(F) = 3$ (Lemma 1), and

$$[3.1]: \quad F \approx x_1x_2x_3. \tag{2.16}$$

2) $m(F) = 5$. Then, $\tilde{\nu}(F) \leq 3$ (the above 1) and Lemmas 1, 2 and 4).

2.1) If $\tilde{\nu}(F) = 1$, then $\tilde{m}(F) = 3$.

2.2) $\tilde{\nu}(F) = 2$.

$$[5.2]: \quad F \approx (x_1x_2 + x_3x_4)x_5. \tag{2.17}$$

2.3) $\tilde{\nu}(F) = 3$. It follows from Lemmas 3 and 4 and the above 1) that

$$\begin{aligned} F &\approx x_2x_3x_4 + x_5[(x_1 + a_{13}x_3 + a_{14}x_4)(x_2 + a_{23}x_3 + a_{24}x_4) + x_3x_4] \\ &\approx (x_2 + a_{23}x_3 + a_{24}x_4)x_3x_4 + x_5[(x_1 + a_{13}x_2 + a_{14}x_4)(x_2 + a_{23}x_3 + a_{24}x_4) + x_3x_4] \\ &\approx x_2x_3x_4 + x_5(x_1x_2 + x_3x_4) \\ &\approx (x_2 + x_5)x_3x_4 + x_5x_1x_2 \\ &\approx (x_2 + x_5)x_3x_4 + (x_2 + x_5)x_1x_2 \\ &\approx (x_2 + x_5)(x_1x_2 + x_3x_4) \\ &\approx x_5(x_1x_2 + x_3x_4). \end{aligned} \tag{2.18}$$

Hence, $\tilde{\nu}(F) \neq 3$.

3) $m(F) = 6$. It follows from Lemmas 1, 2 and 4 and the above 2) that

$$\tilde{\nu}(F) \leq 4.$$

3.1) $\tilde{\nu}(F) = 2$.

$$[6.2]: \quad F \approx x_1x_2x_3 + x_4x_5x_6. \tag{2.19}$$

3.2) $\tilde{\nu}(F) = 3$. Let $F = F_1 + F_2 + F_3$, where $F_i \in M_{6,3}$. There are two cases:

3.2.1) $\#(L(F_1) \cap L(F_2)) = 1$. It follows from 2) and Lemma 3 that

$$F \approx x_5(x_1x_2 + x_3x_4) + x_6f_1f_2, \tag{2.20}$$

where $f_i = \sum_{j=1}^{5} a_{ij}x_j$ with $1 \leq i \leq 2$ are linearly independent. If f_1 (or f_2) $= x_5$, then $\tilde{\nu}(F) \leq 2$ by Lemma 3. Without loss of generality, we can assume that

$$a_{11} = 1 \quad \text{and} \quad a_{21} = 0. \tag{2.21}$$

Then,

$$\begin{aligned} F &\approx x_5[(x_1 + \sum_{j=2}^{5} a_{1j}x_j)x_2 + (x_3 + a_{14}x_2)(x_4 + a_{13}x_2)] + x_6(x_1 + \sum_{j=2}^{5} a_{1j}x_j)f_2 \\ &\approx x_5(x_1x_2 + x_3x_4) + x_6x_1\sum_{i=2}^{5} b_ix_i. \end{aligned} \tag{2.22}$$

3.2.1.1) If $b_2 = 1$, then

$$\begin{aligned} F &\approx x_5[x_1(x_2 + \sum_{i=3}^{5} b_i x_i) + (x_3 + b_4 x_1)(x_4 + b_3 x_1)] + x_6 x_1(x_2 + \sum_{i=3}^{5} b_i x_i) \\ &\approx (x_5 + x_6)x_1 x_2 + x_5 x_3 x_4. \end{aligned} \tag{2.23}$$

Hence $\tilde{\nu}(F) < 3$. Otherwise, there remain two cases.

3.2.1.2) $b_2 = 0$ and $b_3 = 1$.

$$\begin{aligned} F &\approx x_5[x_1 x_2 + (x_3 + b_4 x_4 + b_5 x_5)x_4] + x_6 x_1(x_3 + b_4 x_4 + b_5 x_5) \\ &\approx x_5(x_1 x_2 + x_3 x_4) + x_6 x_1 x_3. \end{aligned} \tag{2.24}$$

$$[6.3\text{-}1]: \quad F \approx (x_1 x_2 + x_3 x_4)x_5 + x_1 x_3 x_6. \tag{2.25}$$

3.2.1.3) $b_2 = b_3 = b_4 = 0$ and $b_5 = 1$. Then, $\tilde{\nu}(F) \leq 2$ by Lemma 2.

3.2.2) $\#(L(F_i) \cap L(F_j)) = 0$, for $1 \leq i < j \leq 3$. Then,

$$F \approx x_1 x_2 x_3 + x_4 x_5 x_6 + f_1 f_2 f_3, \tag{2.26}$$

where for $1 \leq i \leq 3$, $f_i = g_i + h_i$, $g_i = \sum_{j=1}^{3} a_{ij} x_j$, $h_i = \sum_{j=4}^{6} a_{ij} x_j$, $\{g_1, g_2, g_3\}$ and $\{h_1, h_2, h_3\}$ are linearly independent, respectively. Then, $x_1 x_2 x_3 \approx g_1 g_2 g_3$ and $x_4 x_5 x_6 \approx h_1 h_2 h_3$, respectively, by Lemma 1. Consequently,

$$[6.3\text{-}2]: \quad F \approx x_1 x_2 x_3 + x_4 x_5 x_6 + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6). \tag{2.27}$$

We can check that $\tilde{\nu}(F) = 3$.

3.3) $\tilde{\nu}(F) = 4$. It follows from Lemmas 2 to 4 and the above 2) that

$$F \approx x_5(x_1 x_2 + x_3 x_4) + x_6(f_1 f_2 + f_3 f_4), \tag{2.28}$$

where $f_i = \sum_{j=1}^{5} a_{ij} x_j$ for $1 \leq i \leq 4$ and f_1, f_2, f_3 and f_4 are linearly independent. For $1 \leq i \leq 4$, let $g_i \stackrel{\triangle}{=} \sum_{j=1}^{4} a_{ij} x_j$ and $b_i \stackrel{\triangle}{=} a_{i5}$. If one of b_i's, say b_1, is equal to one, then from Lemma 3, we can assume that $b_i = 0$ for $2 \leq i \leq 4$. Then, $l(g_2, g_3, g_4) = 3$.

3.3.1) Assume that $b_1 = 1$ and g_1 is a linear sum $\sum_{i=2}^{4} c_i g_i$. Note that

$$[(f_1 + \sum_{i=2}^{4} c_i f_i)f_2 + (f_3 + c_4 f_2)(f_4 + c_3 f_2)] + f_1 f_2 + f_3 f_4 \in \bar{M}_{5,1}. \tag{2.29}$$

Then

$$F \approx x_5(x_1 x_2 + x_3 x_4 + x_6 g_2) + x_6 g_3 g_4, \tag{2.30}$$

and from Lemma 3, $\tilde{\nu}(F) \leq 3$.

3.3.2) Assume that $l(g_1, g_2, g_3, g_4) = 4$. Without loss of generality, assume that $a_{11} = 1$ and $a_{i1} = 0$ for $2 \leq i \leq 4$ (from Lemma 3). By symmetry, there are two cases:

3.3.2.1) $a_{22} = 1$.

From Lemma 3, $a_{i2} = 0$ for $3 \leq i \leq 4$, and

$$\begin{aligned} F &\approx x_5(x_1x_2 + x_3x_4) + x_6[f_1f_2 + (x_3 + a_{35}x_5)(x_4 + a_{45}x_5)] \\ &\approx x_5[x_1x_2 + (x_3 + a_{35}x_5)(x_4 + a_{45}x_5)] + x_6[f_1f_2 + (x_3 + a_{35}x_5)(x_4 + a_{45}x_5)] \\ &\approx (x_5 + x_6)x_3x_4 + x_5x_1x_2 + x_6f_1f_2, \end{aligned} \tag{2.31}$$

where $f_1 = x_1 + a_{13}x_3 + a_{14}x_4 + a_{15}x_5$ and $f_2 = x_2 + a_{23}x_3 + a_{24}x_4 + a_{25}x_5$. Hence, $\tilde{\nu}(F) \leq 3$.

3.3.2.2) $a_{22} = 0$.

Without loss of generality, $a_{23} = 1$. From Lemma 3,

$$\begin{aligned} F &\approx x_5(x_1x_2 + x_3x_4) + x_6[(x_1 + a_{12}x_2 + \sum_{j=4}^{5} a_{1j}x_j)(x_3 + \sum_{j=4}^{5} a_{2j}x_j) \\ &\quad + (x_2 + a_{35}x_5)(x_4 + a_{45}x_5)] \qquad (2.32) \\ &\approx x_5[(x_1 + a_{12}x_2 + \sum_{j=4}^{5} a_{1j}x_j)(x_2 + a_{35}x_5) + (x_3 + a_{14}x_2 + a_{24}x_4 + a_{25}x_5)(x_4 + a_{45}x_5)] \\ &\quad + x_6[(x_1 + a_{12}x_2 + \sum_{j=4}^{5} a_{1j}x_j)(x_3 + \sum_{j=4}^{5} a_{2j}x_j) + (x_2 + a_{35}x_5)(x_4 + a_{45}x_5)] \\ &\approx x_5(x_1x_2 + x_3x_4) + x_6[x_1(x_3 + a_{14}x_2) + x_2x_4] \qquad (2.33) \\ &\approx x_5[x_1(x_2 + x_3) + x_3(x_1 + x_4)] + x_6[x_1(x_2 + x_3) + x_2(x_4 + \bar{a}_{14}x_1)] \\ &\approx (x_5 + x_6)x_1(x_2 + x_3) + x_5x_3(x_1 + x_4) + x_6x_2(x_4 + \bar{a}_{14}x_1). \qquad (2.34) \end{aligned}$$

Hence, $\tilde{\nu}(F) \leq 3$.

Table 1 summarize the above results. For $m = 6$, there are exactly five representative leaders of $\mathrm{RM}_{6,3}/\mathrm{RM}_{6,2}$ other than 0. For each equivalent block of cosets other than $\mathrm{RM}_{6,2}$, the leader, the minimum weight $w^{(6)}$ of the cosets in each block and the number of codewords with the minimum weight, denoted $A_{w^{(6)}}$, in the cosets are listed. Note that the weight of the binary 2^6-tuple represented by the leader is equal to the minimum weight of the coset. $\Delta\Delta$

Next, the case of $m(F) = 7$ is considered.

Example 2: Consider the case of $m(F) = 7$.

4) It follows from Lemmas 1, 2 and 4 and the above 3) in Example 1 that

$$3 \le \tilde{\nu}(F) \le 6. \tag{2.35}$$

4.1) $\tilde{\nu}(F) = 3$. Let $F = F_1 + F_2 + F_3 \in \bar{M}_{7,3}$, where $F_i \in M_{7,3}$ for $1 \le i \le 3$.
4.1.1) $\#(L(F_1) \cap L(F_2)) = 1$.

$$F \approx x_5(x_1x_2 + x_3x_4) + x_6x_7f, \tag{2.36}$$

where $f = \sum_{i=1}^{5} a_ix_i$. There are two cases.
4.1.1.1) $a_i = 0$ for $1 \le i \le 4$ and $a_5 = 1$.

$$[7.3\text{-}1]: \quad F \approx (x_1x_2 + x_3x_4 + x_5x_6)x_7. \tag{2.37}$$

4.1.1.2) $a_1 = 1$.

$$F \approx x_5[(x_1 + \sum_{i=2}^{5} a_ix_i)x_2 + (x_3 + a_4x_2)(x_4 + a_3x_2)] + x_6x_7f, \tag{2.38}$$

$$[7.3\text{-}2]: \quad F \approx (x_1x_2 + x_3x_4)x_5 + x_1x_6x_7. \tag{2.39}$$

4.1.2) $\#(L(F_i) \cap L(F_j)) = 0$ for $1 \le i < j \le 3$.

$$F \approx x_1x_2x_3 + x_4x_5x_6 + x_7f_1f_2, \tag{2.40}$$

where for $1 \le i \le 2$, $f_i = g_i + h_i$, $g_i = \sum_{j=1}^{3} a_{ij}x_j$, $h_i = \sum_{j=4}^{6} a_{ij}x_j$, $\{g_1, g_2\}$ and $\{h_1, h_2\}$ are linearly independent, respectively. By Lemma 1-(1),

$$[7.3\text{-}3]: \quad F \approx x_1x_2x_3 + x_4x_5x_6 + (x_1 + x_4)(x_2 + x_5)x_7. \tag{2.41}$$

4.2) $\tilde{\nu}(F) \ge 4$. Consider a representative of $e(F)$ with the form of $F_1 + x_7f$, where $F_1 \in \bar{M}_{6,3}$ and $f \in M_{6,2}$. Let $\tilde{(\nu)}_1(F) \stackrel{\triangle}{=} \nu(F_1)$ and $\tilde{\nu}_2(F) = \nu(f)$. By Lemma 2, $1 \le \tilde{\nu}_2(F) \le 3$. We choose a representative of $e(F)$ with smallest $\tilde{m}(F)$, $\tilde{\nu}(F)$ and $\tilde{\nu}_2(F)$ in this order as stated after Lemma 4.
4.2.1) $\tilde{\nu}_1(F) = 1$ and $\tilde{\nu}_2(F) = 3$.

$$F \approx x_1x_2x_3 + x_7(f_1f_2 + f_3f_4 + f_5f_6). \tag{2.42}$$

There are two cases:

4.2.1.1) $F \approx x_1x_2x_3 + x_7(x_4g_1 + x_5g_2 + x_6g_3)$.

4.2.1.2) $F \approx x_1x_2x_3 + x_7(x_4x_5 + x_6g_1 + g_2g_3)$.

By Lemma 3 (or operations on g_i's), we can assume that there is no occurrence of x_6 in g_i's. By exchanging x_7 for x_6, both 4.2.1.1) and 4.2.1.2) are reduced to the case of $\tilde{\nu}_1(F) = 3$ and $\tilde{\nu}_2(F) = 1$.

4.2.2) $\tilde{\nu}_1(F) = 2$ and $\tilde{\nu}_2(F) \geq 2$.

By 2.2), 3.1) and Lemmas 3 and 4, it is sufficient to consider the following two cases 4.2.2.1) and 4.2.2.2).

4.2.2.1) $F \approx (x_1x_2 + x_3x_4)x_5 + x_7(x_6f_2 + \cdots + f_{2h-1}f_{2h})$, for $2 \leq h \leq 3$, where $f_i = \sum_{j=1}^{5} a_{ij}x_j$ for $2 \leq i \leq 2h$ and f_i's are linearly independent. By $x_6 \leftrightarrow x_7$, this case is reduced to the case of $\tilde{\nu}_1(F) \leq 3$ and $\tilde{\nu}_2(F) = 1$.

4.2.2.2) $F \approx x_1x_2x_3 + x_4x_5x_6 + x_7(f_1f_2 + \cdots + f_{2h-1}f_{2h})$, for $2 \leq h \leq 3$, where $f_i = g_i + h_i$, $g_i = \sum_{j=1}^{3} a_{ij}x_j$, $h_i = \sum_{j=4}^{6} b_{ij}x_j$, and $l(f_1, f_2, \cdots, f_{2h}) = 2h$ for $1 \leq i \leq 2h$. If $\tilde{\nu}_2(F) = h = 3$, then from Lemma 3, we can assume that $f_{2h} = x_6$. By $x_6 \leftrightarrow x_7$, this case is reduced to the case of $\tilde{\nu}_1(F) \leq 3$ and $\tilde{\nu}_2(F) = 2$. Suppose that $\tilde{\nu}_2(F) = h = 2$. If $l(x_1, x_2, x_3, f_1, f_2, f_3, f_4) < 6$ or $l(x_4, x_5, x_6, f_1, f_2, f_3, f_4) < 6$, then this case is reduced to the case of $\tilde{\nu}_1(F) \leq 3$ and $\tilde{\nu}_2(F) = 1$. Assume that $l(x_1, x_2, x_3, f_1, f_2, f_3, f_4) = l(x_4, x_5, x_6, f_1, f_2, f_3, f_4) = 6$. Then, $l(g_1, g_2, g_3, g_4) = l(h_1, h_2, h_3, h_4) = 3$. Without loss of generality, assume that $l(g_1, g_2, g_3) = 3$ and $g_4 = \sum_{j=1}^{3} a_jg_j$. Since

$$f_1f_2 + f_3f_4 \approx (f_1 + a_2f_3)(f_2 + a_1f_3) + f_3(f_4 + a_1f_1 + a_2f_2 + a_3f_3), \tag{2.43}$$

we can assume that $g_4 = 0$. Then, h_4 is not a constant and $\sum_{j=1}^{4} b_jh_j =$ a constant, where b_1, b_2, b_3, b_4 are not all zero. By symmetry, it is sufficient to consider two cases: $b_3 = 0$ and $b_1 = 1$ or $b_3 = 1$. For the first case, since $f_1f_2 + f_3f_4 \approx (f_1 + b_2f_2 + b_4f_4)f_2 + (f_3 + b_4f_2)f_4$, we can assume that $h_1 = 0$. Since $x_1x_2x_3 \approx g_1g_2g_3$ and $x_4x_5x_6 \approx h_2h_3h_4$, we can assume that

$$\begin{cases} g_i = x_i, & \text{for} \quad 1 \leq i \leq 3, \\ h_i = x_{i+2}, & \text{for} \quad 2 \leq i \leq 4, \\ g_4 = h_1 = 0. \end{cases} \tag{2.44}$$

Then,

$$F \approx x_1x_2x_3 + x_4x_5x_6 + x_7(x_1(x_2 + x_4) + (x_3 + x_5)x_6)$$

$$\approx \quad x_1(x_2x_3 + (x_2 + x_4)x_7) + x_6(x_4x_5 + (x_3 + x_5)x_7)$$
$$\approx \quad x_5(x_2x_1 + x_4x_3) + x_7((x_4 + x_2)x_6 + (x_1 + x_6)x_3). \tag{2.45}$$

This case is reduced to case 4.2.2.1).

For the second case, since $f_1f_2 + f_3f_4 \approx (f_1 + b_2f_4)(f_2 + b_1f_4) + (f_3 + b_1f_1 + b_2f_2 + b_4f_4)f_4$, we can assume that $h_3 = 0$. Since $x_1x_2x_3 \approx g_1g_2g_3$ and $x_4x_5x_6 \approx h_1h_2h_4$, we can assume that

$$\begin{cases} g_i = x_i, & \text{for} \quad 1 \le i \le 3, \\ h_i = x_{i+3}, & \text{for} \quad 1 \le i \le 2, \\ h_4 = x_6, \\ g_4 = h_3 = 0. \end{cases} \tag{2.46}$$

Then,

$$F \approx x_1x_2x_3 + x_4x_5x_6 + x_7((x_1 + x_4)(x_2 + x_5) + x_3x_6). \tag{2.47}$$

The weight of the binary 2^7-tuple $\mathbf{v}$ represented by the right-hand side of (2.47) is 40, while the minimum weight of the coset $\mathbf{v} + \mathrm{RM}_{7,2}$ is 36. By computer search, we found the following representative of weight 36.

$$[7.4.2]: \quad F \approx x_1x_2x_3 + x_4x_5x_6 + x_2x_3 + x_4x_6 + ((x_1 + x_4)(x_2 + x_5) + x_3x_6)x_7. \tag{2.48}$$

4.2.3) $\tilde{\nu}_1(F) = 3$ and $\tilde{\nu}_2(F) \ge 1$.

From Lemma 4 and case 3) of Example 1, there remains two cases:

$$F \approx F_1 + x_7(f_1f_2 + \cdots + f_{2h-1}f_{2h}), \tag{2.49}$$

where $1 \le h \le 3$, $l(f_1, f_2, \cdots, f_{2h}) = 2h$ and

$$F_1 = G \stackrel{\triangle}{=} (x_1x_2 + x_3x_4)x_5 + x_1x_3x_6, \tag{2.50}$$

or

$$F_1 = H \stackrel{\triangle}{=} x_1x_2x_3 + x_4x_5x_6 + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6). \tag{2.51}$$

These two cases are analyzed by using 1) invariant linear transformations of G or H and 2) the following fact: A polynomial F need not be chosen as a new representative if we can show the existence of an $F' \approx F$ such that $\nu_2(F') \le \nu_2(F)$ and $F'_{x_7=0} \not\approx G$ or H. It is shown in Appendices G and H that there remain the following two representatives:

$$[7.4.1\text{-}1]: \quad F \approx G + x_2x_3x_7, \tag{2.52}$$
$$[7.4.1\text{-}2]: \quad F \approx H + x_1x_4x_7. \tag{2.53}$$

Consequently, there are exactly 12 equivalent blocks of cosets in $\mathrm{RM}_{7,3}/\mathrm{RM}_{7,2}$. Table 2 lists the representative leaders of equivalent blocks other than 0, the minimum weight $w^{(7)}$ of the cosets in each block and the number of codewords with the minimum weight, denoted $A_{w^{(7)}}$ in a coset of each block. Note that the weight of the binary 2^7-tuple represented by the leader is equal to the minimum weight of the coset. In Table 3, the weight distribution of a coset in each block is listed. The representative leader 0 is labeled by [0.0] in the table. Since the number of vectors with weight w in a coset is equal to that with weight $128 - w$, the numbers of vectors with weight not greater than 64 is presented in the table. The number of vectors in any coset with unshown weight not greater than 64 is 0.

The results obtained here can also be used for constructing linear unequal error protection (LUEP) codes [8]. By adding leaders of cosets whose linear sums have minimum weights greater than that of $\mathrm{RM}_{m,2}$ to the generator matrix of $\mathrm{RM}_{m,2}$, we can obtain a generator matrix of an LUEP code of length 2^m. For example, from Table 5, we see that there are cosets with minimum weight 40. By using it, we obtain a $(128, 30)$ LUEP code. Codewords in correspondence to 1 information bit are at a Hamming distance at least 40, and codewords in correspondence to 29 information bits are at a Hamming distance at least 32. $\Delta\Delta$

3. Extended codes of permuted cyclic codes which are super codes of $\mathrm{RM}_{7,2}$ and subcodes of $RM_{7,3}$

Let $\mathrm{CRM}_{m,r}$ be the r-th order cyclic Reed-Muller code of length $2^m - 1$. Let S_u be the cyclotomic coset modulo $2^7 - 1$ over GF(2) which contains u, and let α be a primitive element of $\mathrm{GF}(2^7)$. Then, the set of zeros of $\mathrm{CRM}_{7,3}$ is

$$\{\alpha^i : i \in Z_{7,3}\}, \tag{3.1}$$

where

$$Z_{7,3} \stackrel{\triangle}{=} S_1 \cup S_3 \cup S_5 \cup S_7 \cup S_9 \cup S_{11} \cup S_{13} \cup S_{19} \cup S_{21}. \tag{3.2}$$

The set of zeros of $\mathrm{CRM}_{7,2}$ is

$$\{\alpha^i : i \in Z_{7,2}\}, \tag{3.3}$$

where

$$Z_{7,2} \stackrel{\triangle}{=} Z_{7,3} \cup S_{15} \cup S_{23} \cup S_{27} \cup S_{29} \cup S_{43}. \tag{3.4}$$

Let $C_u^{(\mathrm{irr})}$ be the irreducible cyclic code of length 2^7-1 whose nonzeros are $\{\alpha^j : j \in S_u\}$. Then, any cyclic code C with $\mathrm{CRM}_{7,2} \subseteq C \subseteq \mathrm{CRM}_{7,3}$ can be represented as the following direct sum:

$$\sum_{u \in U} C_u^{(\mathrm{irr})} + \mathrm{CRM}_{7,2}, \tag{3.5}$$

where

$$U \subseteq \{15, 23, 27, 29, 43\}. \tag{3.6}$$

This cyclic code is denoted by C_U.

For a permutation π on $\{1, 2, \ldots, 2^7\}$ and a 2^7-tuple $\mathbf{u} = (u_1, \ldots, u_{2^7})$, let $\pi(\mathbf{u})$ denote the 2^7-tuple $\mathbf{u} = (u_{\pi(1)}, \ldots, u_{\pi(2^7)})$. For a code D of length 2^7 and a permutation π, let $\pi[D]$ denote $\{\pi(\mathbf{v}) : \mathbf{v} \in D\}$. For two codes D_1 and D_2 of the same length, D_1 and D_2 are said to be equivalent if and only if there is a permutation π such that $D_2 = \pi[D_1]$. Equivalent codes have the same weight distribution. Consider the permutation π_0 on $\{1, 2, \ldots, 2^7\}$ such that $\pi_0[\mathrm{exCRM}_{7,r}] = \mathrm{RM}_{7,r}$, where $\mathrm{exCRM}_{7,r}$ is the extended code of $\mathrm{CRM}_{7,r}$. Let $\mathrm{ex}C_U$ denote the extended code of C_U. Then,

$$\mathrm{RM}_{7,2} \subseteq \pi_0[\mathrm{ex}C_U] \subseteq \mathrm{RM}_{7,3}.$$

(1) Consider the case where $|U| = 1$. Since $C_u^{(\mathrm{irr})}$ with $u \neq 0$ is the simplex code, $\pi_0[\mathrm{ex}C_U]/\mathrm{RM}_{7,2}$ consists of two equivalent blocks. The leader of one block is 0. The representative leader of the other block is listed in Table 4, for every $U \subseteq \{15, 23, 27, 29, 43\}$ with $|U| = 1$.

$\mathrm{ex}C_{\{43\}}$ is the extended binary primitive $(128, 36)$ BCH code, and $\mathrm{ex}C_{\{15\}}$ is the dual code of the extended binary primitive $(128, 92)$ BCH code. From Table 4, these two codes and $\mathrm{ex}C_{\{27\}}$ are equivalent, and $\mathrm{ex}C_{\{29\}}$ and $\mathrm{ex}C_{\{23\}}$ are also equivalent, but these two sets of equivalent codes are not equivalent. These codes are LUEP codes, and the (128,36)BCH code was known as an LUEP code [9]. For these codes, we see from the table that the separation vector is $(36, 32)$ for $k_1 = 7$, $k_2 = 29$. That is, codewords in correspondence to 7 information bits are at a Hamming distance at least 36, and codewords in correspondence to 29 information bits are at a Hamming distance at least 32.

(2) Consider the case where $|U| = 2$. The 2^{14} cosets of $\pi_0[\mathrm{ex}C_U]/\mathrm{RM}_{7,2}$ are partitioned into equivalent blocks for every $U \subseteq \{15, 23, 27, 29, 43\}$ with $|U| = 2$. $\mathrm{ex}C_{\{43,29\}}$ is the extended binary primitive $(128, 43)$ BCH code, and $\mathrm{ex}C_{\{29,15\}}$ is the dual code of the extended binary

primitive $(128, 85)$ BCH code. The number of cosets in each equivalent block other than a block containing $\mathrm{RM}_{7,2}$ is presented in Table 5. Since 127 is prime, the number of cosets in each equivalent block is a multiple of 127 except for the block containing $\mathrm{RM}_{7,2}$. There are only 3 different weight distributions for these 10 codes. From tha table, we see that the extended $(128, 43)$ BCH code and the dual code of the extended $(128, 85)$ BCH code have the same weight distribution [4].

(3) Consider the case where $|U| = 3$. The 2^{21} cosets of $\pi_0[\mathrm{ex}C_U]/\mathrm{RM}_{7,2}$ are partitioned into equivalent blocks for $U = \{43, 29, 27\}$ and $U = \{29, 23, 15\}$. $\mathrm{ex}C_{\{43,29,27\}}$ is the extended binary primitive $(128, 50)$ BCH code, and $\mathrm{ex}C_{\{29,23,15\}}$ is the dual code of the extended binary primitive $(128, 78)$ BCH code. The number of cosets in each equivalent block is presented in Table 5.

4. Conclusion

In this paper, cosets of the second order Reed-Muller code in the third order Reed-Muller code are partitioned into equivalent classes. Two cosets are in the same block, if and only if there is a transformation in the general linear group by which one coset is transformed into the other. We show that there are 12 blocks for the case where the code length is 2^7. The representative coset leader of each block is presented. By using the result, several cyclic codes have been shown to be equvalent. Some new LUEP codes have been also constructed. A remaining problem is to devise an efficient algorithm to decide whether given two cosets are equivalent or not.

References

[1] Kasami, T., Takata, T., Fujiwara, T. and Lin S., "On Complexity of Trellis Structure of Linear Block Codes," *IEEE Trans. Inf. Theory*, vol. 39, no. 3, pp. 1057–1064, May 1993.

[2] Desaki, Y., Fujiwara, T. and Kasami, T., "A Method for Computing the Weight Distribution of a Block Code by Using Its Trellis Diagram," *IEICE Trans. Fundamentals*, vol. E77-A, no. 8, pp.1230–1237, 1994.

[3] Fujiwara, T., Kasami, T. and Desaki, Y., "A New Method for Computing the Weight Distribution of a Linear Block Code Based on Its Trellis Structure," *IEICE Technical Report*, IT94-10, May 1994.

[4] Fujiwara, T., Kasami, T. and Desaki, Y., "New Methods for Computing the Weight Distribution of a Linear Block Code Based on Its Trellis Structure," *Proc. of the International Symposium on Information Theory and Its Applications*, pp.25–28, Sydney, Australia, Nov. 1994.

[5] Desaki, Y., Fujiwara, T. and Kasami, T., "A Method for Computing the Weight Distribution of a Linear Block Code by Using Weight Distributions of Cosets with Respect to Its Subcode," *Proc. of the 17th Symposium on Information Theory and Its Applications*, pp.213–216, Hiroshima, Japan, Dec. 1994.

[6] Peterson, W.W. and Weldon, E.J., Jr., Error-Correcting Codes, MIT Press, 1972.

[7] MacWilliams, F.J. and Sloane, N.J.A., *The Theory of Error-Correcting Codes*, North-Holland, 1977.

[8] Masnick, B. and Wolf, J., "On Linear Unequal Error Protection Codes," *IEEE Trans. Info. Theory*, vol.IT-13, no.4, pp.600–607, Oct. 1967.

[9] Morelos-Zaragoza, R.H. and Lin S., "On Primitive BCH Codes With Unequal Error Protection Capabilities," to appear in the IEEE Transactions on Information Theory, vol.41, no.3, May 1995.

Table 1: The leader of each equivalent block of cosets $\mathrm{RM}_{6,3}/\mathrm{RM}_{6,2}$ other than $\mathrm{RM}_{6,2}$, the minimum weight $w^{(6)}$ of the cosets in each block and the number of vectors with the minimum weight, denoted $A_{w^{(6)}}$, in the cosets.

label	representative polynomial	$w^{(6)}$	$A_{w^{(6)}}$
[3.1]	$x_1x_2x_3$	8	8
[5.2]	$(x_1x_2 + x_3x_4)x_5$	12	32
[6.2]	$x_1x_2x_3 + x_4x_5x_6$	14	64
[6.3-1]	$(x_1x_2 + x_3x_4)x_5 + x_1x_3x_6$	16	448
[6.3-2]	$x_1x_2x_3 + x_4x_5x_6 + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6)$	18	3584

Table 2: The leader of each equivalent block of cosets $\mathrm{RM}_{7,3}/\mathrm{RM}_{7,2}$ other than $\mathrm{RM}_{7,2}$, the minimum weight $w^{(7)}$ of the cosets in each block and the number of vectors with the minimum weight, denoted $A_{w^{(7)}}$, in the cosets.

label	representative polynomial	$w^{(7)}$	$A_{w^{(7)}}$
[3.1]	$x_1x_2x_3$	16	8
[5.2]	$(x_1x_2 + x_3x_4)x_5$	24	32
[6.2]	$x_1x_2x_3 + x_4x_5x_6$	28	64
[6.3-1]	$(x_1x_2 + x_3x_4)x_5 + x_1x_3x_6$	32	448
[6.3-2]	$x_1x_2x_3 + x_4x_5x_6 + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6)$	36	3584
[7.3-1]	$(x_1x_2 + x_3x_4 + x_5x_6)x_7$	28	128
[7.3-2]	$(x_1x_2 + x_3x_4)x_5 + x_1x_6x_7$	32	128
[7.3-3]	$x_1x_2x_3 + x_4x_5x_6 + (x_1 + x_4)(x_2 + x_5)x_7$	36	768
[7.4.1-1]	$(x_1x_2 + x_3x_4)x_5 + x_1x_3x_6 + x_2x_3x_7$	36	1024
[7.4.1-2]	$x_1x_2x_3 + x_4x_5x_6 + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6) + x_1x_4x_7$	40	11264
[7.4.2]	$x_1x_2x_3 + x_4x_5x_6 + x_2x_3 + x_4x_6 + ((x_1 + x_4)(x_2 + x_5) + x_3x_6)x_7$	36	128

Table 3: Weight Distributions.

weight	[0.0]	[3.1]	[5.2]	[6.2]	[6.3-1]	[6.3-2]
0	1	0	0	0	0	0
16	0	8	0	0	0	0
24	0	0	32	0	0	0
28	0	0	0	64	0	0
32	10668	1680	512	0	448	0
36	0	0	0	3136	0	3584
40	0	71680	48032	14336	25088	0
44	0	0	131072	274176	229376	330240
48	5291328	3205496	2911232	2609152	2822400	2580480
52	0	14680064	15859712	16459520	15826944	16289280
56	112881664	62003200	57119808	55494656	56634880	55738368
60	0	102760448	114032640	117480832	116064256	117594624
64	300503590	171425760	156664832	152199168	153664128	151797760

weight	[7.3-1]	[7.3-2]	[7.3-3]	[7.4.1-1]	[7.4.1-2]	[7.4.2]
28	128	0	0	0	0	0
32	0	128	0	0	0	0
36	128	1024	768	1024	0	128
40	43008	18432	15872	14336	11264	16128
44	161280	261120	274688	273408	307200	279936
48	2838528	2729472	2739200	2781184	2693120	2741760
52	16160256	16118784	16014080	15819776	15986688	15969408
56	55891968	55760896	55878144	56379392	55972864	55931136
60	116847360	117836800	117928192	117074944	117923840	117968256
64	152985600	151417600	151169024	152182784	151080960	151057408

Table 4: The representative polynomial for a codeword in irreducible cyclic code $C_u^{(\mathrm{irr})}$.

u	$w^{(7)}$	$A_{w^{(7)}}$	label	representative polynomial
15	36	128	[7.4.2]	$x_1x_2x_3 + x_4x_5x_6 + x_2x_3 + x_4x_6 + ((x_1 + x_4)(x_2 + x_5) + x_3x_6)x_7$
23	36	1024	[7.4.1-1]	$(x_1x_2 + x_3x_4)x_5 + x_1x_3x_6 + x_2x_3x_7$
27	36	128	[7.4.2]	$x_1x_2x_3 + x_4x_5x_6 + x_2x_3 + x_4x_6 + ((x_1 + x_4)(x_2 + x_5) + x_3x_6)x_7$
29	36	1024	[7.4.1-1]	$(x_1x_2 + x_3x_4)x_5 + x_1x_3x_6 + x_2x_3x_7$
43	36	128	[7.4.2]	$x_1x_2x_3 + x_4x_5x_6 + x_2x_3 + x_4x_6 + ((x_1 + x_4)(x_2 + x_5) + x_3x_6)x_7$

Table 5: Classification of representative polynomials for a vector in the sum of two or three irreducible codes.

label	$w^{(7)}$	$A_{w^{(7)}}$	U				
			{43,29} {43,23} {29,27} {29,15} {27,23} {23,15}	{43,27} {43,15} {27,15}	{29,23}	{43,29,27}	{29,23,15}
[3.1]	16	8	0	0	0	0	0
[5.2]	24	32	0	0	0	0	0
[6.2]	28	64	0	0	1×127	21×127	36×127
[6.3-1]	32	448	0	0	0	35×127	63×127
[6.3-2]	36	3584	0	0	0	7×127	28×127
[7.3-1]	28	128	0	1×127	0	1×127	1×127
[7.3-2]	32	128	7×127	0	0	1071×127	882×127
[7.3-3]	36	768	70×127	84×127	84×127	8540×127	8736×127
[7.4.1-1]	36	1024	2×127	0	2×127	109×127	81×127
[7.4.1-2]	40	11264	0	0	0	217×127	126×127
[7.4.2]	36	128	50×127	44×127	42×127	6512×127	6560×127

Appendix G

Case G where $F_{x_7=0} = (x_1x_2 + x_3x_4)x_5 + x_1x_3x_6$

Lemma G.1: Define

$$G \stackrel{\triangle}{=} (x_1x_2 + x_3x_4)x_5 + x_1x_3x_6, \tag{G.1}$$

The following are invariant linear transformations of G.

(1.1) $x_1 \leftrightarrow x_3, \quad x_2 \leftrightarrow x_4,$

(1.2) $x_1 \leftrightarrow x_5, \quad x_4 \leftrightarrow x_6,$

(1.3) $x_3 \leftrightarrow x_5, \quad x_2 \leftrightarrow x_6.$

(2) $x_2 \leftarrow a_1x_1+x_2+a_3x_3+a_4x_4+a_5x_5+a_6x_6, \quad x_3 \leftarrow a_4x_1+x_3+a_6x_5, \quad x_4 \leftarrow a_3x_1+x_4.$

(3) $x_1 \leftarrow x_1 + a_6x_5, \quad x_4 \leftarrow a_1x_1 + a_3x_3 + x_4 + a_5x_5 + a_6x_6, \quad x_6 \leftarrow a_1x_5 + x_6.$

(4) $x_6 \leftarrow a_1x_1 + a_3x_3 + x_6.$

(5) $x_1 \leftarrow x_1 + a_3x_3 + a_5x_5, \quad x_4 \leftarrow a_3x_2 + x_4 + a_5x_6.$

(5') $x_3 \leftarrow a_1x_1 + x_3 + a_5x_5, \quad x_2 \leftarrow x_2 + a_1x_4 + a_5x_6.$

(5") $x_5 \leftarrow a_1x_1 + a_3x_3 + x_5, \quad x_6 \leftarrow a_3x_2 + a_1x_4 + x_6.$ $\Delta\Delta$

Now we consider the following case:

$$F \approx G + x_7(f_1f_2 + \cdots + f_{2h-1}f_{2h}), \quad 1 \le h \le 3, \tag{G.2}$$

where G is defined by (G.1), and $f_i = \sum_{j=1}^{6} a_{ij}x_j$ for $1 \le i \le 2h$ and $l(f_1, \cdots, f_{2h}) = 2h$.

(a) There is an index i such that $1 \le i \le 3$ and

$$a_{2i-1,2j} = a_{2i,2j} = 0, \quad \text{for} \quad 1 \le j \le 3. \tag{G.3}$$

By (1.1) to (1.3) of Lemma G.1, we can assume that

$$a_{2i-1,1} = 1, \quad a_{2i,1} = 0 \quad \text{and} \quad a_{2i,3} = 1. \tag{G.4}$$

Without loss of generality, $i = 1$. By (5) and (5') of Lemma G.1, we can assume that

$$f_1 = x_1 \quad \text{and} \quad f_2 = x_3. \tag{G.5}$$

Hence, by $x_6 \leftarrow x_6 + x_7$, $\tilde{m}(F) \le 6$ or $\tilde{\nu}_2(F) < h$.

($\bar{\text{a}}$) For $1 \leq i \leq h$, there is an even index j such that

$$a_{2i,j} = 1. \tag{G.6}$$

By transformations (1.1) to (1.3) and (2) in Lemma G.1 which keep the above condition (G.6), we can assume that

$$f_2 = x_2, \tag{G.7}$$

and that if there is an index i such that

$$a_{2i-1,2} = a_{2i-1,4} = a_{2i-1,6} = 0, \quad \text{for} \quad 1 \leq i \leq h, \tag{G.8}$$

then

$$a_{1,2} = a_{1,4} = a_{1,6} = 0. \tag{G.9}$$

By linear operations on f_i with $i \neq 2$ which keep $f_1 f_2 + \cdots + f_{2h-1} f_{2h}$ invariant, we can assume that

$$a_{i,2} = 0, \quad \text{for} \quad i \neq 2. \tag{G.10}$$

Then,

$$F \approx G' + x_2(x_1 x_5 + x_7 f_1), \tag{G.11}$$

$$G' \approx x_3 x_4 x_5 + x_1 x_3 x_6 + x_7(f_3 f_4 + \cdots), \tag{G.12}$$

where there is no occurrence of x_2 in G' and therefore, $\tilde{\nu}(G') \leq 3$. Hence, (α) if $a_{1,3}$, $a_{1,4}$ or $a_{1,6}$ is one, then

$$\tilde{\nu}_2(F) \leq 2 \tag{G.13}$$

and ($\bar{\alpha}$) otherwise,

$$\tilde{\nu}_2(F) \leq 1. \tag{G.14}$$

From 3.2) in Example 1, there are two cases of G':

$$G' \approx G, \tag{G.15}$$

or

$$G' \approx H \stackrel{\triangle}{=} x_1 x_2 x_3 + x_4 x_5 x_6 + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6). \tag{G.16}$$

The latter case is studied in Appendix H.

(b) First consider the former case where $h = 1$ and $(\bar{\alpha})$: $a_{1,3} = a_{1,4} = a_{1,6} = 0$. Then, by (1.2) of Lemma G.1, $a_{1,1}$ can be assumed to be one and

$$F \approx (x_1 + a_{1,5}x_5)x_2(x_5 + x_7) + x_3x_4x_5 + x_1x_3x_6, \tag{G.17}$$

and therefore, $\tilde{\nu}(F) \leq 3$.

Next consider the remaining case where $h \leq 2$ and $a_{1,3}$, $a_{1,4}$ or $a_{1,6}$ is one.

(c) Condition (G.8) holds.

(c1) $a_{1,1}$ (or $a_{1,5}$) $= 1$. By (5) (or (5")) of Lemma G.1,

$$f_1 = x_1 \quad (\text{or} \quad x_5). \tag{G.18}$$

Hence, this case is reduced to (b).

(c$\bar{1}$) $f_1 = x_3$.

For $h = 1$,

$$[7.4.1\text{-}1] : \quad F \approx G + x_2x_3x_7. \tag{G.19}$$

For $h = 2$, by (1.2) and (3) of Lemma G.1, we can assume that

$$f_4 = x_4. \tag{G.20}$$

Then,

$$F \approx G'' + x_4(x_3x_5 + x_7f_3), \tag{G.21}$$

$$G'' \approx x_1x_2x_5 + x_1x_3x_6 + x_2x_3x_7, \tag{G.22}$$

where there is no occurrence of x_4 in G''. Since $f_4 = x_4$, we can assume that

$$a_{3,4} = 0. \tag{G.23}$$

If $a_{31} = a_{36} = 0$, then it follows from (G.10) that $l(x_3, x_5, x_7, f_3) < 4$ and $h = 1$. It is sufficient to consider the case where

$$a_{3,1} \quad \text{or} \quad a_{3,6} = 1. \tag{G.24}$$

(d) $a_{3,6} = 1$.

By (G.10), (G.23) and Lemma G.1-(4),

$$f_3 = a_{3,5}x_5 + x_6. \tag{G.25}$$

$$
\begin{aligned}
F &\approx (x_1(x_2+a_{3,5}x_3)+x_3x_4)x_5+x_1x_3(a_{3,5}x_5+x_6)+((x_2+a_{3,5}x_3)x_3+x_4(a_{3,5}x_5+x_6))x_7\\
&\approx G+(x_2x_3+x_4x_6)x_7\\
&\approx G+[(x_2+x_4)x_3+x_4(x_3+x_6)]x_7\\
&\approx [(x_1+x_3)x_2+x_3(x_2+x_4)]x_5+(x_1+x_3)x_3x_6+[(x_2+x_4)x_3+x_4(x_3+x_6)]x_7\\
&\approx (x_2x_5+x_3(x_3+x_6))(x_1+x_3)+(x_2+x_4)x_3(x_5+x_7)+x_4(x_3+x_6)x_7\\
&\approx x_1x_2x_3+x_4x_5x_6+x_7((x_1+x_4)(x_2+x_5)+x_3x_6),\\
&\quad \text{by}\quad x_1\leftarrow x_2+x_4, x_2\leftarrow x_5+x_7, x_3\leftarrow x_3, x_4\leftarrow x_4, x_5\leftarrow x_7, x_6\leftarrow x_3+x_6\\
&\quad \text{and}\quad x_7\leftarrow x_1+x_3 (=(2.47)\approx [7.4.2]). \qquad \text{(G.26)}
\end{aligned}
$$

($\bar{\text{d}}$) $a_{3,1}=1$, $a_{3,6}=0$.

If $a_{3,5}=1$, then by (5") of Lemma G.1, we can assume that

$$f_3=x_5. \qquad \text{(G.27)}$$

Since $F=(x_1x_2+x_3x_4)x_5+x_1x_3x_6+x_7(x_2x_3+x_4x_5)\approx (x_1x_2+(x_3+x_7)x_4)x_5+x_2(x_3+x_7)x_3+x_1x_3x_6$, $\tilde{\nu}(F)\leq 4$ and $\tilde{\nu}_2(F)\leq 1$. Therefore, we can assume that

$$a_{3,5}=0. \qquad \text{(G.28)}$$

Then,

$$f_3=x_1+a_{3,3}x_3. \qquad \text{(G.29)}$$

$$F\approx G+(x_2x_3+(x_1+a_{3,3}x_3)x_4)x_7. \qquad \text{(G.30)}$$

$$
\begin{aligned}
F &\approx [(x_1+x_3)x_2+x_3(x_2+x_4)]x_5+x_1x_3x_6+[(x_2+x_4)x_3+(x_1+ax_3)x_4]x_7, a\in\{0,1\},\\
&\approx (x_2+x_4)x_3(x_5+x_7)+(x_1+x_3)x_2x_5+(x_1+ax_3)x_4x_7+x_1x_3x_6. \qquad \text{(G.31)}
\end{aligned}
$$

(i) $a=1$.

$$
\begin{aligned}
F &\approx [x_2(x_5+x_7)+(x_2+x_4)x_7](x_1+x_3)+(x_2+x_4)x_3(x_5+x_7)+(x_1+x_3)x_3x_6\\
&\approx G+x_2x_3x_7=[7.4.1\text{-}1]. \qquad \text{(G.32)}
\end{aligned}
$$

(ii) $a=0$.

$$F\approx x_1x_2x_3+x_4x_5x_6+(x_1+x_4)(x_2+x_5)(x_3+x_6)+x_1x_4x_7=[7.4.1\text{-}2]. \qquad \text{(G.33)}$$

($\bar{\text{c}}$) $a_{1,4}$ or $a_{1,6} = 1$ and if $h = 2$, $a_{3,4}$ or $a_{3,6} = 1$. By (1.2) and (3) of Lemma G.1, we can assume that

$$a_{1,4} = 1, \tag{G.34}$$

and

$$f_1 = x_4. \tag{G.35}$$

($f_2 = x_2$ is not changed.)

For $h = 1$,

$$F \approx G + x_2x_4x_7. \tag{G.36}$$

By $x_7 \leftarrow x_5 + x_7$ and $x_6 \leftarrow x_5 + x_6$,

$$F \approx \bar{F} \stackrel{\triangle}{=} (x_1x_2 + x_3x_4)x_5 + x_1x_3(x_5 + x_6) + x_2x_4(x_5 + x_7).$$

$$\begin{aligned} \bar{F}_{x_7=0} &\approx (x_1x_2 + x_3x_4)x_5 + (x_1x_3 + x_2x_4)x_5 + x_1x_3x_6 \\ &\approx x_3(x_1 + x_4)x_5 + x_2(x_1 + x_4)x_5 + x_1x_3x_6 \\ &\approx (x_2 + x_3)(x_1 + x_4)x_5 + x_1x_3x_6 \\ &\not\approx G \text{ or } H. \end{aligned} \tag{G.37}$$

For $h = 2$, do the following operations. If $a_{3,6} + a_{4,6} = 0$, $f_3 + f_4 \rightarrow f_3$ and if $a_{3,6} = 1$ and $a_{4,6} = 0$, then $f_3 \leftrightarrow f_4$ and then, if $a_{3,4} = 1$, $f_1 + f_3 \rightarrow f_3$ and $f_2 + f_4 \rightarrow f_2$. Then, this case is reduced to case (c) where $a_{32} = a_{34} = a_{36} = 0$.

Appendix H

H1: Lemmas for Case H where $F_{x_7=0} = x_1x_2x_3 + x_4x_5x_6 + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6)$

A polynomial F such that $F_{x_7=0} = H$ is not chosen as a representative if we can show the existence of an $F' \approx F$ such that $\nu_2(F') \leq \nu_2(F)$ and $F'_{x_7=0} \not\approx H$.

Lemma H.1: The following are invariant linear transformations of $H \stackrel{\triangle}{=} x_1x_2x_3 + x_4x_5x_6 + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6)$.

(1.1) $x_1 \leftrightarrow x_2, \quad x_4 \leftrightarrow x_5,$

(1.2) $x_2 \leftrightarrow x_3, \quad x_5 \leftrightarrow x_6,$

(1.3) $x_3 \leftrightarrow x_1, \quad x_6 \leftrightarrow x_4,$

(1.4) $x_1 \leftrightarrow x_4, \quad x_2 \leftrightarrow x_5, \quad x_3 \leftrightarrow x_6.$

(2) $x_i \leftarrow \sum_{j=1}^{3} a_{ij}x_j$, $x_{i+3} \leftarrow \sum_{j=1}^{3} a_{ij}x_{j+3}$ for $1 \leq i \leq 3$, where the coefficient matrix (a_{ij}) is regular.

(3) $x_i \leftarrow x_i + x_{i+3}$ for $1 \leq i \leq 3$.

Lemma H.2: Let

$$F \stackrel{\triangle}{=} H + x_7(f_1f_2 + \cdots + f_{2h-1}f_{2h}), \quad 2 \leq h \leq 3, \tag{H1.1}$$

where $f_i = \sum_{j=1}^{6} a_{ij}x_j$ with $1 \leq i \leq 2h$ are linearly independent linear sums of $x_1, x_2, \cdots, x_6$. Let

$$f_1f_2 + \cdots + f_{2h-1}f_{2h} \approx \sum_{1 \leq i < j \leq 6} b_{ij}x_ix_j, \quad b_{ij} \in GF(2). \tag{H1.2}$$

If there is an index i with $1 \leq i \leq 3$ such that $b_{ii+3} = 0$, then $\tilde{\nu}_2(F) \leq 2$.

(Proof) For simplicity, let $i = 1$. Then

$$\begin{aligned} F \quad \approx \quad & x_4x_5x_6 + x_4(x_2 + x_5)(x_3 + x_6) + x_7 \sum_{1 < i < j \leq 6} b_{ij}x_ix_j \\ & + x_1(x_2x_3 + (x_2 + x_5)(x_3 + x_6) + x_7 \sum_{j=2}^{6} b_{1j}x_j). \end{aligned} \tag{H1.3}$$

Iff $b_{14} = 1$, $l(x_2, x_3, x_2 + x_5, x_3 + x_6, x_7, \sum_{j=2}^{2h} b_{1j}x_j) = 6$. $\Delta\Delta$

Remark 1: Let $h = 2$. From Corollary 1, we can assume that

$$f_1 = x_1, \tag{H1.4}$$

$$a_{i1} = 0, \qquad \text{for} \quad 2 \le i \le 4. \tag{H1.5}$$

If $a_{24} = 0$, then

$$F \approx F_1 + x_1(x_2x_3 + (x_2 + x_5)(x_3 + x_6) + x_7f_2), \tag{H1.6}$$

where $F_1 \approx x_4x_5x_6 + x_4(x_2 + x_5)(x_3 + x_6) + x_7f_3f_4$. Then, $F_1 \not\approx H$ and $\nu_2(F) \le 2$.

Remark 2: Suppose that

$$F \approx (x_1x_2 + x_3x_4)x_5 + (f_1f_2 + f_3f_4)x_6, \tag{H1.7}$$

where f_i's $(= \sum_{j=1}^{5} a_{ij}x_j)$ with $1 \le i \le 4$ are linearly independent.

Let $g_i \stackrel{\triangle}{=} \sum_{j=1}^{4} a_{ij}x_j$ for $i \le i \le 4$. From 3.3), we have the following:

(R1) If $l(g_1, g_2, g_3, g_4) < 4$, then $F \not\approx H$ (from 3.3.1) and (2.30)).

(R2) Suppose that $l(g_1, g_2, g_3, g_4) = 4$. Without loss of generality, assume that $a_{11} = 0$ and $a_{i1} = 0$ for $2 \le i \le 4$ (from Lemma 3). By symmetry,

(R2.1) $a_{22} = 1$. Then, $a_{i2} = 0$ for $i = 1, 3, 4$ (from Lemma 3). From (2.31), $F \not\approx H$, if and only if

$$\begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} = 1. \tag{H1.8}$$

(R2.2) $a_{22} = 0$. Without loss of generality, $a_{23} = 1$, and then $a_{13} = a_{33} = a_{43} = 0$ (from Lemma 3). From (2.32) and (2.34), $F \not\approx H$ if and only if $a_{14} = 0$.

(R2.2') As a special case, from (2.34),

$$(x_1x_2 + x_3x_4)x_5 + (x_1x_3 + x_2x_4)x_6 \not\approx H. \tag{H1.9}$$

H2: Case H

Let

$$F \stackrel{\triangle}{=} H + (f_1f_2 + \cdots + f_{2h-1}f_{2h})x_7, \tag{H2.1}$$

where $f_i = \sum_{j=1}^{6} a_{ij}x_j$ for $1 \le i \le 2h$ and $l(f_1, f_2, \cdots, f_{2h}) = 2h$.

By Lemma 3, we can assume that

$$a_{11} = 1, \quad \text{and} \quad a_{i1} = 0, \quad \text{for} \quad 2 \le i \le 2h. \tag{H2.2}$$

Let $x_i' = x_i + x_{i+3}$ for $1 \le i \le 3$. If $h = 3$ and $a_{24} = 0$, then from Lemma H.2, $\tilde{\nu}_2(F) \le 2$.
1) $h = 3$ and $a_{24} = 1$. By Lemma 3, we can assume that

$$a_{14} = 0, \quad \text{and} \quad a_{i4} = 0, \quad \text{for} \quad 3 \le i \le 6. \tag{H2.3}$$

By (1.2) and (2) of Lemma H.1, we can assume that

$$a_{12} = a_{13} = a_{15} = 0. \tag{H2.4}$$

If $a_{16} = 1$, $a_{22} + a_{25} = 0$ and $a_{23} + a_{26} = 1$, then by $f_1 \leftarrow f_1 + f_2$, $x_1 \leftarrow x_1 + a_{22}x_2 + a_{23}x_3$ and $x_4 \leftarrow x_4 + a_{22}x_5 + a_{23}x_6$, we have the following new coefficients:

$$\begin{cases} a_{11} = a_{14} = 1, \quad a_{12} = a_{13} = a_{15} = a_{16} = 0, \\ a_{24} = a_{26} = 1, \quad a_{21} = a_{25} = 0. \end{cases} \tag{H2.5}$$

By the second transformation $x_i \leftarrow x_i'$ with $1 \le i \le 3$, this case is reduced to the case of $a_{16} = 0$. Then there are two cases:
(i) $a_{11} = 1$, $a_{1i} = 0$ for $2 \le i \le 6$, $a_{21} = 0$, $a_{24} = 1$. By applying (1.2) and (2) of Lemma H.1 twice, it can be shown that $a_{23} = 0$ and either (i.1) $a_{25} = 0$ or (i.2) $a_{25} = 1$ and $a_{26} = 0$. That is,

$$f_2 = c_2x_2 + x_4 + c_5x_5 + c_6x_6, \tag{H2.6}$$

where $c_5c_6 = 0$.
(ii) $a_{11} = a_{16} = 1$, $a_{1i} = 0$ for $2 \le i \le 5$, $a_{21} = 0$, $a_{24} = 1$, either $a_{22} + a_{25} = 1$ or $a_{23} + a_{26} = 0$. By (2) of Lemma H.1, it can be shown that there are three cases of $a_{22}, a_{23}, a_{25}, a_{26}$.
(ii.1) $a_{23} = a_{26} = 0$.

$$f_2 = c_2x_2 + x_4 + c_5x_5. \tag{H2.7}$$

(ii.2) $a_{22} = a_{26} = c$, $a_{23} = a_{25} = \bar{c}$.

$$f_2 = cx_2 + \bar{c}x_3 + x_4 + \bar{c}x_5 + cx_6. \tag{H2.8}$$

(ii.3) $a_{22} = a_{25} = 0$, $a_{23} = a_{26} = 1$.

$$f_2 = x_3 + x_4 + x_6. \tag{H2.9}$$

(The remaining cases where $a_{21}=0, a_{24}=1$ and either $a_{22}+a_{25}=1$ or $a_{23}+a_{26}=0$ are (1) $a_{22}=a_{23}=a_{25}=a_{26}=1$, (2) $a_{22}=a_{23}=0$ and $a_{25}=a_{26}=1$ and (3) $a_{22}=a_{23}=1$ and $a_{25}=a_{26}=0$. These cases can be reduced to (ii.3), (ii.1) with $c_2=0$ and $c_5=1$ and (ii.1) with $c_2=1$ and $c_5=0$, respectively, by (2) of Lemma H.1.)

Then f_i's with $3 \le i \le 6$ are linearly independent linear sums of x_2', x_3', x_5 and x_6. Let f_i be expressed as $b_{i2}x_2' + b_{i3}x_3' + b_{i5}x_5 + b_{i6}x_6$. By Lemma 3, we can assume that

$$b_{32}=1, \quad \text{and} \quad b_{i2}=0, \quad \text{for} \quad 4 \le i \le 6. \tag{H2.10}$$

1.1) $b_{43}=1$. By Lemma 3, we can assume that

$$\begin{cases} b_{i3}=0, & \text{for} \quad i=3,5,6, \\ b_{55}=b_{66}=1, & \\ b_{56}=b_{65}=0. & \end{cases} \tag{H2.11}$$

By Lemma H.2, if $b_{45}=0$ or $a_{16}a_{23}+b_{36}=0$, then $\tilde{\nu}_2(F) \le 2$. We can assume that

$$b_{45}=a_{16}a_{23}+b_{36}=1. \tag{H2.12}$$

Then we have that

$$F \approx F_1 + x_1(x_2x_3 + x_2'x_3' + x_7f_2), \tag{H2.13}$$

$$\begin{aligned} F_1 \approx\ & x_2'x_3'x_4 + (x_2' + b_{35}x_5 + b_{36}x_6)(x_3' + x_5 + b_{46}x_6)x_7 \\ & +((x_4+x_7)x_5 + a_{16}x_7f_2)x_6. \end{aligned} \tag{H2.14}$$

We first consider case (i) where $a_{16}=0$, $b_{36}=1$ and $f_2=c_2x_2+x_4+c_5x_5+c_6x_6$ with $c_5c_6=0$. Then, $F_1 \approx H$ only if $l(b_{35}x_5+x_6, x_5+b_{46}x_6)=2$, that is,

$$b_{35}b_{46}=0. \tag{H2.15}$$

Thus we have that

$$\begin{aligned} F \approx\ & H' + x_7(x_1(c_2x_2 + x_4 + c_5x_5 + c_6x_6) \\ +\ & (x_2 + b_5x_5 + x_6)(x_3 + x_5 + b_6x_6)), \end{aligned} \tag{H2.16}$$

(H2.17)

where $b_5 \stackrel{\triangle}{=} \bar{b}_{35}$, $b_6 \stackrel{\triangle}{=} \bar{b}_{46}$ and

$$H' \stackrel{\triangle}{=} x_1x_2x_3 + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6) + (x_4 + x_7)x_5x_6. \tag{H2.18}$$

where $c_5c_6 = 0$ and $\bar{b}_5\bar{b}_6 = 0$. Hereafter, we write $F_{x_7=0}$ instead of $\bar{F}_{x_7=0}$ for a polynomial $\bar{F}$ which is equivalent to F. By $x_7 \leftarrow x_1 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx x_1x_2x_3 + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6) + (x_1 + x_4)x_5x_6 \\ &\quad + x_1(x_2 + b_5x_5 + x_6)(x_3 + x_5 + b_6x_6) \\ &\approx x_1(x_2x_3 + (x_2 + b_5x_5 + x_6)(x_3 + x_5 + b_6x_6)) \\ &\quad + (x_1 + x_4)((x_2 + x_5)(x_3 + x_6) + x_5x_6). \end{aligned} \tag{H2.19}$$

If $b_5 = b_6 = 1$, then $F_{x_7=0} \not\approx H$. Hence we can assume that $b_5b_6 = 0$. Since $\bar{b}_5\bar{b}_6 = 0$, we have that

$$b_6 = \bar{b}_5, \tag{H2.20}$$

and

$$\begin{aligned} & x_2x_3 + (x_2 + bx_5 + x_6)(x_3 + x_5 + \bar{b}x_6) \\ &\approx (x_2 + bx_3 + bx_6)x_5 + (\bar{b}x_2 + x_3 + \bar{b}x_5)x_6 \\ &\approx (x_2 + x_5 + b(x_3 + x_6))x_5 + (\bar{b}(x_2 + x_5) + x_3 + x_6)x_6, \end{aligned} \tag{H2.21}$$

and

$$\begin{aligned} & (x_2 + x_5)(x_3 + x_6) + x_5x_6 \\ &\approx (x_2 + x_5 + b(x_3 + x_6))(\bar{b}(x_2 + x_5) + x_3 + x_6) + x_5x_6. \end{aligned} \tag{H2.22}$$

From (R2.2'), $F_{x_7=0} \not\approx H$. Then we conclude that this case reduces to a previous case.

Next we consider case (ii) where $a_{16} = 1$ and $a_{23} + b_{36} = 1$. Then, we have that

$$\begin{aligned} F \approx\ & H' + x_7((x_1 + x_6)f_2 \\ & + (x_2 + b_5x_5 + \bar{a}x_6)(x_3 + x_5 + b_6x_6)), \end{aligned} \tag{H2.23}$$

where $a = a_{23}$. If $c_2 = a_{22} = a_{23} = 0$, then $\bar{b}_5\bar{b}_6 = 0$ from (H2.13) and (H2.14), this is a necessary condition that $F_1 \approx H$. By $x_7 \leftarrow x_1 + x_6 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx x_1x_2x_3 + (x_1 + x_4)[(x_2 + x_5)(x_3 + x_6) + x_5x_6] \\ &\quad + (x_1 + x_6)(x_2 + b_5x_5 + ax_6)(x_3 + x_5 + b_6x_6). \end{aligned} \tag{H2.24}$$

(A) Suppose $\begin{vmatrix} b_5 & \bar{a} \\ 1 & b_6 \end{vmatrix} = 1$, that is, $b_5 b_6 = a$. Then there is (α, β) such that

$$\alpha(b_5 x_5 + \bar{a} x_6) + \beta(x_5 + b_6 x_6) = x_6, \tag{H2.25}$$

and therefore,

$$\begin{aligned} & x_1 x_2 x_3 + (x_1 + x_6)(x_2 + b_5 x_5 + \bar{a} x_6)(x_3 + x_5 + b_6 x_6) \\ \approx\ & (x_1 + \alpha x_2 + \beta x_3)[x_2 x_3 + (x_2 + b_5 x_5 + \bar{a} x_6)(x_3 + x_5 + b_6 x_6)]. \end{aligned} \tag{H2.26}$$

(A.1) $a = 0$.

If $b_6 = \bar{b}_5$, then it follows from the argument for (i) that $F_{x_7=0} \not\approx H$. Hence we can assume that

$$a_{23} = 0, b_{36} = 1, b_5 = b_6 = 0. \tag{H2.27}$$

Since $\bar{b}_5 \bar{b}_6 = 1$, we can assume that

$$c_2 = a_{22} = 1. \tag{H2.28}$$

Consequently,

$$F \approx H' + x_7((x_1 + x_6)(x_2 + x_4 + c_5 x_5 + c_6 x_6) + (x_2 + x_6)(x_3 + x_5)). \tag{H2.29}$$

By $x_7 \leftarrow x_2 + x_4 + c_5 x_5 + c_6 x_6 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx (x_2 + x_4 + c_5 x_5 + c_6 x_6)((x_2 + x_6)(x_3 + x_5) + x_5 x_6) + H \\ &\approx (x_4 + c_5 x_5 + \bar{c}_6 x_6)(x_2 + x_6)(x_2' + x_3') + (x_2 + x_4) x_5 x_6 + H \\ &\approx (x_2' + x_3')(x_1' x_2' + (x_4 + c_5 x_5 + \bar{c}_6 x_6)(x_2 + x_6)) + x_2(x_1 x_3 + x_5 x_6) \\ &\approx (x_2' + x_3')((x_1 + c_5 x_5 + \bar{c}_6 x_6) x_2' + (x_4 + c_5 x_5 + \bar{c}_6 x_6)(x_5 + x_6)) + x_2(x_1 x_3 + x_5 x_6) \\ &\approx x_4(x_2 + x_3)(x_5 + x_6) + \quad \text{a form of} \quad x_1, x_2, x_3, x_5, x_6 \\ &\not\approx H. \end{aligned} \tag{H2.30}$$

($\overline{\text{A.1}}$) $a = 1$. This case is a special case of the following ($\overline{\text{A}}$).

($\overline{\text{A}}$) $b_5 = b_6 = 1$. Then,

($\overline{\text{A}}$.1) $a = a_{23} = 0$. Then, $b_{36} = 1$, and

$$x_1 x_2 x_3 + (x_1 + x_6)(x_2 + x_5 + x_6)(x_3 + x_5 + x_6)$$

$$\approx (x_2+x_3)[x_1x_3+(x_1+x_6)(x_3+x_5+x_6)], \tag{H2.31}$$

$$(x_1+x_4)[(x_2+x_5)(x_3+x_6)+x_5x_6]$$
$$\approx (x_1+x_4)(x_2+x_3)(x_3+x_6)+(x_1+x_4)(x_5+x_6)x_3. \tag{H2.32}$$

Since $l(x_1, x_3, x_1+x_6, x_3+x_5+x_6, x_1+x_4, x_3+x_6) < 6$, $F_{x_7=0} \not\approx H$.

($\overline{\text{A}}$.2) $a = a_{23} = 1$. Then, $b_{36} = 0$, f_2 is case (ii.2) with $c = 0$ or case (ii.3), and

$$\begin{aligned} F &\approx H' + x_7((x_1+x_6)(x_3+x_4+cx_5+\bar{c}x_6) \\ &\quad +(x_2+x_5)(x_3+x_5+x_6)). \end{aligned} \tag{H2.33}$$

By $x_7 \leftarrow x_3+x_4+cx_5+\bar{c}x_6+x_7$,

$$\begin{aligned} F_{x_7=0} &\approx (x_3+x_4+cx_5+\bar{c}x_6)((x_2+x_5)(x_3+x_5+x_6)+x_5x_6)+H \\ &\approx x_3(x_1x_2+x_5x_6)+x_2'(x_1'x_3'+(x_4+\bar{c}x_5+cx_6)(x_3'+x_5)) \\ &\approx x_4x_2'(x_3'+x_5)+ \quad \text{a form of} \quad x_1, x_2, x_3, x_5, x_6 \\ &\not\approx H. \end{aligned} \tag{H2.34}$$

1.2) $b_{43} = a_{43} = 0$. For this case, we use expressions $f_i = a_{i2}x_2 + a_{i3}x_3 + a_{i5}x_5 + a_{i6}x_6$ for $3 \le i \le 6$. There are two cases: either $a_{45} = 1$ or $a_{45} = 0$ and $a_{46} = 1$.

1.2.1) $a_{45} = 1$. By Lemma 3, we can assume that

$$\begin{cases} a_{i5} = 0, \quad \text{for} \quad i = 3, 5 \quad \text{and} \quad 6, \\ a_{53} = 1, \quad a_{63} = 0, \\ a_{66} = 1. \end{cases} \tag{H2.35}$$

Then we have that

$$\begin{aligned} F &\approx H + x_7((x_1+a_{16}x_6)(c_2x_2+c_3x_3+x_4+c_5x_5+c_6x_6) \\ &\quad +(x_2+a_{33}x_3+a_{36}x_6)(x_5+a_{46}x_6)+x_3x_6). \end{aligned} \tag{H2.36}$$

By $x_7 \leftarrow x_3+x_7$,

$$\begin{aligned} F_{x_7=0} &\approx H+(x_1+a_{16}x_6)x_3(c_2x_2+x_4+c_5x_5+c_6x_6)+(x_2+a_{36}x_6)x_3(x_5+a_{46}x_6) \\ &\approx x_3(x_1(\bar{c}_2x_2+x_4+c_5x_5+c_6x_6)+(x_2+a_{36}x_6)(x_5+a_{46}x_6) \end{aligned}$$

$$
\begin{aligned}
& +(x_1+x_4)(x_2+x_5)+a_{16}x_6(c_2x_2+x_4+c_5x_5))+x_6(x_1'x_2'+x_4x_5) \\
\approx\ & x_3(x_1(c_2x_2+x_4+\bar{c}_5x_5+c_6x_6)+x_4(x_2+x_5+a_{16}x_6) \\
& +(x_2+(c_5a_{16}+a_{36})x_6)(x_5+(c_2a_{16}+a_{46})x_6))+x_6(x_1'x_2'+x_4x_5) \\
\approx\ & x_3((x_1+x_2+x_5+a_{16}x_6)(c_2x_2+x_4+\bar{c}_5x_5+c_6x_6) \\
& +(x_2+x_5+a_{16}x_6)((c_2+\bar{c}_5)x_5+(c_6+c_2a_{16})x_6) \\
& +(x_2+x_5+a_{16}'x_6)(x_5+a_{46}'x_6))+x_6(x_1'x_2'+x_4x_5),
\end{aligned}
\tag{H2.37}
$$

where $a_{16}' \stackrel{\triangle}{=} (c_2+c_5)a_{16}+a_{36}+a_{46}$ and $a_{46}' \stackrel{\triangle}{=} c_2a_{16}+a_{46}$.

$$
\begin{aligned}
F \ \approx\ & x_3((x_1+x_2+x_5+a_{16}x_6)(c_2x_2+x_4+\bar{c}_5x_5+c_6x_6) \\
& +(x_2+x_5+\beta_1x_6)((c_2+c_5)x_5+\beta_2x_6)) \\
& +x_6(x_1'x_2'+((c_2+\bar{c}_5)a_{16}x_3+x_4)x_5),
\end{aligned}
$$

where

$$
\beta_1 \stackrel{\triangle}{=} a_{16}+a_{36}+a_{46} \quad \text{and} \quad \beta_2 \stackrel{\triangle}{=} c_6+a_{46}. \tag{H2.38}
$$

If $c_2=c_5$, then $F_{x_7=0} \not\approx H$ from (R1). Suppose that

$$
c_2+c_5=1. \tag{H2.39}
$$

Then,

$$
\begin{aligned}
F_{x_7=0} \ \approx\ & x_3((x_1+x_2+x_5+a_{16}x_6)(c_2x_2+x_4+c_2x_5+c_6x_6) \\
& +(x_2+x_5+\beta_1x_6)(x_5+\beta_2x_6))+x_6(x_1'x_2'+x_4x_5).
\end{aligned}
\tag{H2.40}
$$

By $x_1 \leftarrow x_1+(\beta_1+a_{16})x_6$, $x_2 \leftarrow x_2+(\beta_1+\beta_2)x_6$, $x_4 \leftarrow x_4+(c_2\beta_1+c_6)x_6$, $x_5 \leftarrow x_5+\beta_2x_6$, the problem that $F_{x_7=0} \not\approx H$ is kept the same. Then,

$$
\begin{aligned}
& x_3((x_1+x_4+\bar{c}_2(x_2+x_5))(x_4+c_2(x_2+x_5))+(x_2+x_5)x_5) \\
& \quad +x_6((x_1+x_4+\bar{c}_2(x_2+x_5))(x_2+x_5)+x_4x_5) \not\approx H,
\end{aligned}
\tag{H2.41}
$$

if $c_2=0$ from (R2.2'). We can assume that

$$
c_2=1 \quad \text{and} \quad c_5=0. \tag{H2.42}
$$

By $x_7 \leftarrow x_6 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx x_6(x_1(x_2 + c_3x_3 + x_4) + (x_2 + a_{33}x_3)x_5 + x_4x_5 + x_1'x_2') \\ &\quad + x_3(x_1x_2 + x_1'x_2') \\ &\approx x_6(x_1(c_3x_3 + x_4 + x_5) + (x_2 + a_{33}x_3 + x_4)x_5 + x_4(x_2 + x_5)) \\ &\quad + x_3(x_1x_2 + x_1'x_2') \\ &\approx x_6((x_1 + x_2)(x_4 + x_5) + x_3(c_3x_1 + a_{33}x_5)) \\ &\quad + x_3(x_1x_2 + x_1'x_2') \not\approx H \quad \text{from (R1).} \end{aligned} \tag{H2.43}$$

2) $h = 2$. From Corollary 2 (indices and notations are changed here), we can assume that

$$\begin{cases} f_1 &= \sum_{j=1}^{6} a_{1j}x_j, \\ f_2 &= h_1, \\ f_3 &= g_2 + h_2, \\ f_4 &= g_3 + h_3, \end{cases} \tag{H2.44}$$

where $h_i = \sum_{j=4}^{6} b_{ij}x_j$. Let $g_1 \stackrel{\triangle}{=} \sum_{j=4}^{6} b_{1j}x_{j-3}$.

2.1) $l(g_1, g_2, g_3) = 3$.

By (2) of Lemma H.1, we can assume that

$$g_2 = x_2, g_3 = x_3, h_1 = x_4, a_{14} = 0. \tag{H2.45}$$

From Remark 1 and (1.4) of Lemma H.1,

$$a_{11} = 1. \tag{H2.46}$$

Since $f_1x_4 + (x_2 + h_2)(x_3 + h_3) \approx (x_1 + b_{24}(x_3 + h_3 + b_{34}x_4) + b_{34}(x_2 + h_2 + b_{24}x_4))x_4 + (x_2 + h_2 + b_{24}x_4)(x_3 + h_3 + b_{34}x_4)$, we can assume that

$$b_{24} = b_{34} = 0. \tag{H2.47}$$

By (1.2) and (2) of Lemma H.1 and $f_4 \leftarrow a_{12}f_3 + f_4$, we can assume that

$$a_{12} = 0. \tag{H2.48}$$

By (3) of Lemma H.1 and $f_1 \leftarrow f_1 + b_{25}f_2$, we can also assume that

$$b_{25} = 0. \tag{H2.49}$$

$$F \approx F_2 + x_2'(x_1x_3 + x_1'x_3' + x_7(x_3 + h_3)), \tag{H2.50}$$

$$F_2 \approx x_5(x_1x_3 + x_4x_6) + x_7(f_1x_4 + (x_3 + h_3)h_2'), \tag{H2.51}$$

where $h_2' \stackrel{\triangle}{=} h_2 + x_5 = x_5 + b_{26}x_6$.

(a) Suppose that $b_{35} = 0$. Then $l(x_1, x_3, x_1', x_3', x_7, x_3 + h_3) < 6$. If $b_{26} = 0$, then $F_2 \not\approx H$ from 3.3.1) in Example 1. We can assume that

$$b_{26} = 1. \tag{H2.52}$$

Variables $x_1, x_3, x_4, x_6, x_5, x_7$ in F_2 correspond to $x_1, x_2, x_3, x_4, x_5, x_6$ in (2.32), respectively. Hence, from (R2.2'), $F_2 \not\approx H$ if $a_{16} = 0$.

Consequently, it is sufficient to consider the following case.

$$\text{either} \quad b_{35} = 1, \tag{H2.53}$$

$$\text{or} \quad a_{16} = 1, b_{26} = 1, b_{35} = 0. \tag{H2.54}$$

By (1.2) of Lemma H.1 and the symmetry of x_2 and x_3 in (H2.51), we can assume

$$\text{either} \quad b_{26} = 1, \tag{H2.55}$$

$$\text{or} \quad a_{15} = 1, b_{26} = 0, b_{35} = 1. \tag{H2.56}$$

If $b_{26} = b_{35} = 1$, then by $f_3 \leftarrow f_3 + f_4$, $x_2 \leftarrow x_2 + x_3$, $x_5 \leftarrow x_5 + x_6$ and (3) of Lemma H.1 again, this case is reduced to the case of $b_{26} = 0$. Consequently, either $a_{15} = 1$, $b_{26} = 0$, $b_{35} = 1$ or $a_{16} = 1$, $b_{26} = 1$, $b_{35} = 0$. Then we have that

$$F \approx H + x_7((x_1 + a_{13}x_3 + x_5 + a_{16}x_6)x_4 + x_2(x_3 + x_5 + b_{36}x_6)), \tag{H2.57}$$

$$F \approx H + x_7((x_1 + a_{13}x_3 + a_{15}x_5 + x_6)x_4 + (x_2 + x_6)(x_3 + b_{36}x_6)), \tag{H2.58}$$

where in the second relation one of a_{13} and b_{36} is one in the above form, because otherwise, (H2.58) is a special case of (H2.57) by (1.2) of Lemma H.1.

2.1.1) For (H2.57), by $x_7 \leftarrow x_1 + a_{13}x_3 + ax_4 + x_5 + a_{16}x_6 + x_7$,

$$F_{x_7=0} \approx H + (x_1 + a_{13}x_3 + ax_4 + x_5 + a_{16}x_6)x_2(x_3 + x_5 + b_{36}x_6). \tag{H2.59}$$

2.1.1.1) If $a_{16} = b_{36}$, then let $a = 0$. We have that

$$F_{x_7=0} \approx (x_1 + \bar{a}_{13}x_3)x_2(x_5 + b_{36}x_6) + x_4x_5x_6 + x_1'x_2'x_3', \tag{H2.60}$$

and therefore, $F_{x_7=0} \not\approx H$.

2.1.1.2) If $a_{16} = 1$ and $b_{36} = 0$, then let $a = 1$. Then

$$\begin{aligned} F_{x_7=0} &\approx (x'_1 + a_{13}x'_2 + x'_3)(x_2(x_3 + x_5) + x'_2x'_3) + x_1x_2x_3 + x_4x_5x_6 \\ &\approx (x'_1 + a_{13}x'_2 + x'_3)(x_2x_3 + x'_2(x'_3 + x_5)) + x_1x_2x_3 + x_4x_5x_6 \\ &\approx (x_4 + a_{13}x_5 + x_6)x_2x_3 + (x'_1 + x'_3)x'_2(x'_3 + x_5) + x_4x_5x_6 \\ &\not\approx H. \end{aligned} \tag{H2.61}$$

2.1.1.3) If $a_{13} = a_{16} = 0$ and $b_{36} = 1$, then let $a = 1$. We have that

$$F_{x_7=0} \approx (x'_1 + x'_2)(x'_2 + x'_3)x_5 + x_4x_5x_6 + x_1x_2x_3 \not\approx H. \tag{H2.62}$$

2.1.1.4) If $a_{13} = b_{36} = 1$ and $a_{16} = 0$, then let $a = 0$. Then,

$$\begin{aligned} F_{x_7=0} &\approx (x'_2 + x'_3)(x_2(x_1 + x_3 + x_5) + x'_1x'_2) + x_1x_2x_3 + x_4x_5x_6 \\ &\approx (x'_2 + x'_3)(x_2(x_1 + x_3) + (x_1 + x_4 + x_5)x'_2) + (x_1 + x_3)x_2x_3 + x_4x_5x_6 \\ &\approx (x_5 + x_6)x_2(x_1 + x_3) + (x'_1 + x_5)x'_2x'_3 + x_4x_5x_6 \\ &\not\approx H. \end{aligned} \tag{H2.63}$$

2.1.2) For (H2.58), by $x_7 \leftarrow x_1 + a_{13}x_3 + a_{15}x_5 + x_6 + x_7$ where $\bar{a}_{13}\bar{b}_{36} = 0$,

$$F_{x_7=0} \approx H + (x_1 + a_{13}x_3 + a_{15}x_5 + x_6)(x_2 + x_6)(x_3 + b_{36}x_6). \tag{H2.64}$$

2.1.2.1) If $b_{36} = 1$, then

$$\begin{aligned} F_{x_7=0} &\approx (x_2 + x_3)((x_1 + a_{13}x_3 + a_{15}x_5 + x_6)(x_3 + x_6) + x_1x_3) + x'_1x'_2x'_3 + x_4x_5x_6 \\ &\approx (x_2 + x_3)((a_{15}x_5 + x_6)(x_3 + x_6) + (x_1 + a_{13}x_3)x_6) + x'_1x'_2x'_3 + x_4x_5x_6. \end{aligned} \tag{H2.65}$$

2.1.2.1.1) If $a_{15} = 1$, then

$$\begin{aligned} F_{x_7=0} &\approx (x_1 + x_4 + x_5 + x_6)(x'_2 + x'_3)x'_3 + x_6((x_2 + x_3)(x_1 + a_{13}x_3) + x_4x_5) \\ &\not\approx H. \end{aligned} \tag{H2.66}$$

2.1.2.1.2) If $a_{15} = 0$, then

$$\begin{aligned} F_{x_7=0} &\approx (x_2 + x_3)x_6(x_1 + \bar{a}_{13}x_3) + x'_1x'_2x'_3 + x_4x_5x_6 \\ &\not\approx H. \end{aligned} \tag{H2.67}$$

2.1.2.2) If $b_{36} = 0$, then

$$\begin{aligned} F_{x_7=0} &\approx H + (x_1 + a_{15}x_5 + x_6)(x_2 + x_6)x_3 \\ &\approx x_3(x_1x_2 + (x_1 + a_{15}x_5 + x_6)(x_2 + x_6)) + x'_1x'_2x'_3 + x_4x_5x_6 \\ &\approx x_3(x_1x_6 + (a_{15}x_5 + x_6)(x_2 + x_6)) + x'_1x'_2x'_3 + x_4x_5x_6. \end{aligned} \tag{H2.68}$$

If $a_{15} = 0$, then $F_{x_7=0} \not\approx H$. If $a_{15} = 1$, then

$$\begin{aligned} F_{x_7=0} &\approx x_6(x_1x_3 + x_4x_5) + x'_2(x'_1x'_3 + x_3(x_5 + x_6)) \\ &\approx x_6(x_1x_3 + x_4x_5 + x'_1x'_2) + (x'_1 + x_5 + x_6)x'_2x_3 \\ &\not\approx H. \end{aligned} \tag{H2.69}$$

2.2) $l(g_1, g_2, g_3) = 2$.
Since g_1 is not a constant, we can assume that

$$g_2 = x_2, \tag{H2.70}$$

$$g_3 = a_1x_1 + a_2x_2. \tag{H2.71}$$

By $f_4 \leftarrow f_4 + a_2f_3$, it is sufficient to consider the following case:

$$g_3 = ax_1. \tag{H2.72}$$

It is easy to see that (H2.44) to (H2.47) can be used except for the second equality of (H2.45).
2.2.1) $g_3 = 0$.
(H2.50) and (H2.51) are modified as follows:

$$F \approx F'_2 + x'_2(x_1x_3 + x'_1x'_3 + x_7(a_{12}x_4 + h_3)), \tag{H2.73}$$

$$F'_2 \approx x_5(x_1x_3 + x_4x_6) + x_7((f_1 + a_{12}x'_2)x_4 + h'_2h_3), \tag{H2.74}$$

where $h'_2 = \bar{b}_{25}x_5 + b_{26}x_6$. If $l(h'_2, h_3) < 2$, then $F'_2 \not\approx H$. If $l(h'_2, h_2) = 2$, then $h'_2h_3 \approx x_5x_6$, and therefore $F'_2 \approx H$ from 3.3.1) in Example refexample:1. If $b_{35} = 0$, $l(x_1, x_3, x'_1, x'_3, x_7, h_3) < 6$ and this case is reduced to a previous case. Hence,

$$b_{35} = 1. \tag{H2.75}$$

F can be rewritten as

$$\begin{aligned} F &\approx F_3 + x_3'(x_1x_2 + x_1'x_2' + a_{13}x_4x_7), && \text{(H2.76)} \\ F_3 &\approx x_6(x_1x_2 + x_4x_5) + x_7(f_1'x_4 + (x_2 + h_2)h_3), && \text{(H2.77)} \end{aligned}$$

where $f_1' = x_1 + a_{12}x_2 + a_{15}x_5 + (a_{13} + a_{16})x_6$. Note that $l(x_1, x_2, x_1', x_2', x_4, x_7) < 6$. By comparing (H2.77) with (2.32), $F_3 \not\approx H$ if $a_{15} = 0$. Then we can assume that

$$a_{15} = 1. \qquad \text{(H2.78)}$$

If $a_{12} = 1$, then by $f_1 \leftarrow f_1 + f_2$, $b_{25} \leftarrow \bar{b}_{25}$, $a_{16} \leftarrow a_{13} + a_{16}$ and (3) of Lemma H.1, a_{15} can be reduced to 0. Hence, we can assume that $a_{12} = 0$. By $f_3 \leftarrow f_3 + b_{25}f_4$, we can assume that $b_{25} = 0$. Consequently we have that

$$F \approx H + x_7((x_1 + a_{13}x_3 + x_5 + a_{16}x_6)x_4 + (x_2 + b_{26}x_6)(x_5 + b_{36}x_6)). \qquad \text{(H2.79)}$$

By $x_7 \leftarrow x_4 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx H + x_4(x_2 + b_{26}x_6)(x_5 + b_{36}x_6) \\ &\approx x_4(x_2 + \bar{b}_{26}x_6)(x_5 + b_{36}x_6) + x_1x_2x_3 + x_1'x_2'x_3'. && \text{(H2.80)} \end{aligned}$$

If $b_{26} = 1$ or $b_{26} = \bar{b}_{36}$, then $F_{x_7=0} \not\approx H$. Hence, we can assume that

$$b_{26} = b_{36} = 0. \qquad \text{(H2.81)}$$

By $x_7 \leftarrow x_5 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx H + (x_1 + a_{13}x_3 + a_{16}x_6)x_4x_5 \\ &\approx (x_1 + a_{13}x_3 + \bar{a}_{16}x_6)x_4x_5 + x_1x_2x_3 + x_1'x_2'x_3'. && \text{(H2.82)} \end{aligned}$$

If $\bar{a}_{16} = a_{13}$, then

$$(x_1 + a_{13}x_3 + \bar{a}_{16}x_6)x_4x_5 \approx (x_1' + a_{13}x_3')x_4x_5, \qquad \text{(H2.83)}$$

and therefore, $F_{x_7=0} \not\approx H$. If $a_{16} = 1$, then $F_{x_7=0} \not\approx H$. Consequently, we can assume that

$$a_{13} = a_{16} = 0. \qquad \text{(H2.84)}$$

$$F \approx H + x_7((x_1 + x_5)x_4 + x_2x_5). \qquad \text{(H2.85)}$$

By $x_3 \leftarrow x_3 + x_7$, F can be transformed into the following form F':

$$\begin{aligned} F' &\approx H + x_7(x_1x_2 + x_1'x_2') + x_7((x_1 + x_5)x_4 + x_2x_5) \\ &\approx H + x_7(x_1x_5 + x_2x_4 + x_1x_4 + x_2x_5) \\ &\approx H + x_7(x_1 + x_2)(x_4 + x_5). \end{aligned} \tag{H2.86}$$

By (2) of Lemma H.2 ($x_1 \leftarrow x_1 + x_2, x_4 \leftarrow x_4 + x_5$),

$$F' \approx H + x_1x_4x_7 = [7.4.1\text{-}2]. \tag{H2.87}$$

2.2.2) $g_3 = x_1$.

$$\begin{aligned} f_1f_2 + f_3f_4 &\approx (f_4 + a_{12}f_2)(f_3 + f_2) + (f_1 + a_{12}f_3 + f_4)f_2 \\ &\approx (x_1 + a_{12}x_4 + b_{35}x_5 + b_{36}x_6)(x_2 + x_4 + b_{25}x_5 + b_{26}x_6) \\ &\quad +(a_{13}x_3 + a_{15}x_5 + a_{16}x_6)x_4, \end{aligned} \tag{H2.88}$$

where $a_{15} \leftarrow a_{15} + b_{35}$ and $a_{16} \leftarrow a_{16} + a_{36}$. By (3) of Lemma H.1, we can assume that $b_{25} = 0$. (H2.76) and (H2.77) are modified for this case as follows:

$$F \approx F_3' + x_3'(x_1x_2 + x_1'x_2' + a_{13}x_4x_7), \tag{H2.89}$$

$$\begin{aligned} F_3' &\approx x_6(x_1x_2 + x_4x_5) \\ &\quad +x_7((x_1 + a_{12}x_4 + b_{35}x_5 + b_{36}x_6)(x_2 + x_4 + b_{26}x_6) + x_4(a_{15}x_5 + a_{16}'x_6)), \end{aligned} \tag{H2.90}$$

where $a_{16}' = a_{13} + a_{16}$. Then, $l(x_1, x_2, x_1', x_2', x_4, x_7) < 6$. If $a_{15} = 0$, then $F_3' \not\approx H$ from 3.3.1) in Example 1. If $a_{15} = 1$, then

$$\begin{aligned} F_3' &\approx (x_6 + x_7)x_4(x_5 + a_{16}'x_6) + x_6x_1x_2 \\ &\quad +x_7(x_1 + a_{12}x_4 + b_{35}x_5 + b_{36}x_6)(x_2 + x_4 + b_{26}x_6). \end{aligned} \tag{H2.91}$$

Then, $F_3' \not\approx H$ iff $b_{35} = 0$. Hence we can assume that

$$a_{15} = b_{35} = 1. \tag{H2.92}$$

We have that

$$\begin{aligned} F &\approx H + x_7((x_1 + a_{12}x_4 + x_5 + b_{36}x_6)(x_2 + x_4 + b_{26}x_6) \\ &\quad +(a_{13}x_3 + x_5 + a_{16}x_6)x_4). \end{aligned} \tag{H2.93}$$

By $x_7 \leftarrow x_4 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx H + (x_1 + x_5 + b_{36}x_6)(x_2 + b_{26}x_6)x_4 \\ &\approx x'_1(x_2 + b_{26}x_6)x_4 + (x_2 + \bar{b}_{26}x_6)x_4(x_5 + b_{36}x_6) \\ &\quad + x'_1(x_2x_3 + x'_2x'_3) + x_2x_3x_4 \\ &\approx x'_1(x_2x_3 + x'_2x'_3 + (x_2 + b_{26}x_6)x_4) \\ &\quad + x_4(x_2x_3 + (x_2 + \bar{b}_{26}x_6)(x_5 + b_{36}x_6)). \end{aligned} \tag{H2.94}$$

If $b_{26} = 1$, then $F_{x_7=0} \not\approx H$. Otherwise,

$$\begin{aligned} F_{x_7=0} &\approx x'_1(x_2x_3 + x'_2x'_3) + x_4(x_2(x'_1 + x'_3 + x_5 + \bar{b}_{36}x_6) + x_5x_6) \\ &\approx x'_1(x_2x_6 + (x'_1 + x'_3 + x_5)x_5) + x_4(x_2(x'_3 + x_5 + \bar{b}_{36}x_6 + x'_1) + x_5x_6). \end{aligned} \tag{H2.95}$$

If $b_{36} = 1$, then $F_{x_7=0} \not\approx H$ from (R2.2'). Consequently we can assume that

$$b_{26} = b_{36} = 0. \tag{H2.96}$$

By $x_7 \leftarrow x_2 + x_4 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx H + x_2(a_{13}x_3 + x_5 + a_{16}x_6)x_4 \\ &\approx x_4(x_5x_6 + x'_2x'_3 + x_2(a_{13}x_3 + x_5 + a_{16}x_6)) + x_1(x_2x_3 + x'_2x'_3) \\ &\approx x_4(x'_2(\bar{a}_{13}x'_3 + x_5 + a'_{16}x_6) + x_5(a_{13}x'_3 + \bar{a}'_{16}x_6)) + x_1(x_2x_6 + x'_3x_5), \end{aligned} \tag{H2.97}$$

where $a'_{16} = a_{13} + a_{16}$. If $a_{13} = 1$ and $a'_{16} = 0$, then $F_{x_7=0} \not\approx H$. If $a_{13} = a'_{16} = 1$, then

$$\begin{aligned} F_{x_7=0} &\approx (x_2 + x_6)(x_4(x_5 + x_6) + x_1x_6) + (x_1 + x_4)x'_3x_5 \\ &\not\approx H. \end{aligned} \tag{H2.98}$$

Thus we can assume that

$$a_{13} = b_{26} = b_{36} = 0. \tag{H2.99}$$

Then,

$$\begin{aligned} F_{x_7=0} &\approx x_4(x'_2(x_5 + a_{16}x_6) + \bar{a}_{16}x_5x_6) + x_1x_2x_3 + x'_1x'_2x'_3 \\ &\not\approx H, \quad \text{if} \quad a_{16} = 1. \end{aligned} \tag{H2.100}$$

We can assume that

$$a_{16} = 0. \tag{H2.101}$$

By $x_7 \leftarrow x_4 + x_5 + x_7$,

$$F_{x_7=0} \approx H + (x_1 + \bar{a}_{12}x_4)(x_2 + x_5)(x_4 + x_5). \tag{H2.102}$$

If $a_{12} = 0$, then

$$\begin{aligned} F_{x_7=0} &\approx (x_4 + x_5)(x_1'x_2' + x_5x_6) + x_1x_2x_3 \\ &\not\approx H. \end{aligned} \tag{H2.103}$$

Consequently, we have that

$$F \approx H + x_7((x_1 + x_4 + x_5)(x_2 + x_4) + x_4x_5), \tag{H2.104}$$

By $x_3 \leftarrow x_3 + x_7$, F is transformed into the following form F':

$$\begin{aligned} F' &\approx H + x_7(x_1x_5 + x_2x_4 + x_4x_5 + x_1x_2 + x_1x_4 + x_2x_4 + x_2x_5) \\ &\approx H + x_7((x_1 + x_5)(x_2 + x_4) + x_1x_5) \\ &\approx H + x_7(x_1 + x_5)(x_2 + x_4 + x_5). \end{aligned} \tag{H2.105}$$

By $x_1 \leftarrow x_1 + x_5$ and $x_4 \leftarrow x_2 + x_4 + x_5$, F is transformed into the following form F':

$$\begin{aligned} F' &\approx (x_1 + x_5)(x_2x_3 + x_2'x_3') + (x_4 + x_2')(x_5x_6 + x_2'x_3') + x_1x_4x_7 \\ &\approx H + x_1x_4x_7 + x_5(x_2x_3 + x_2'x_3') + x_2'x_5x_6 \\ &\approx H + x_1x_4x_7 \\ &\approx [7.4.1\text{-}2]. \end{aligned} \tag{H2.106}$$

2.3) $l(g_1, g_2, g_3) = 1$.

Since $g_1 = x_4$ from (H2.45), g_2 (or g_3) is g_1 or a constant. If g_2 and g_3 are constants, then

$$\begin{aligned} F &\approx H + x_7(f_1x_4 + x_5x_6) \\ &\approx x_1x_2x_3 + x_1'x_2'x_3' + (x_4 + x_7)(x_5x_6 + f_1x_7). \end{aligned} \tag{H2.107}$$

F reduces to case 4.2.2.2) in Example 2. Without loss of generality, consider the case of $g_3 = x_1$. If $g_2 = x_1$, then by $f_3 \leftarrow f_3 + f_4$, we can assume that $g_2 = 0$. Then, (H2.88) is

modified for this case as follows:

$$\begin{aligned} f_1f_2+f_3f_4 &\approx f_1(f_2+f_3)+(f_1+f_4)f_3 \\ &\approx (x_1+a_{12}x_2+a_{13}x_3+a_{15}x_5+a_{16}x_6)(x_4+b_{25}x_5+b_{26}x_6) \\ &\quad +(a_{12}x_2+a_{13}x_3+b_{35}x_5+b_{36}x_6)(b_{25}x_5+b_{26}x_6), \end{aligned} \tag{H2.108}$$

where $b_{35} \leftarrow a_{15}+b_{35}$ and $b_{36} \leftarrow a_{16}+b_{36}$. By (1.2) and (2) of Lemma H.1, we can assume that $a_{12}=0$. If $a_{13}=0$, then this case reduces to that of $h=1$ or $(f_1+f_4)f_3 \approx x_5x_6$. For the second case,

$$F \approx x_1x_2x_3+x_1'x_2'x_3'+(x_4+b_{25}x_5+b_{26}x_6+x_7)(x_5x_6+f_1x_7), \tag{H2.109}$$

and F reduces to case 4.2.2.2) in Example 2. Thus we can assume that

$$a_{13}=1. \tag{H2.110}$$

By $x_1 \leftarrow x_1+x_3$ and $x_4 \leftarrow x_4+x_6$, we can assume that

$$\begin{cases} f_1' = f_1 = x_1+a_{15}x_5+a_{16}x_6, \\ f_2' \stackrel{\triangle}{=} f_2+f_3 = x_4+b_{25}x_5+\bar{b}_{26}x_6, \\ f_3' \stackrel{\triangle}{=} f_1+f_4 = x_3+b_{35}x_5+b_{36}x_6, \\ f_4' \stackrel{\triangle}{=} f_3 = b_{25}x_5+b_{26}x_6. \end{cases} \tag{H2.111}$$

F can be written as

$$F \approx F_2''+x_2'(x_1x_3+x_1'x_3'), \tag{H2.112}$$

$$\begin{aligned} F_2'' &\approx x_5(x_1x_3+x_4x_6)+x_7((x_1+a_{15}x_5+a_{16}x_6)(x_4+b_{25}x_5+\bar{b}_{26}x_6) \\ &\quad +(x_3+b_{35}x_5+b_{36}x_6)(b_{25}x_5+b_{26}x_6)). \end{aligned} \tag{H2.113}$$

If $b_{26}=0$, then $F_2'' \not\approx H$ from 3.3.1) in Example 1. We can assume that

$$b_{26}=1. \tag{H2.114}$$

From (R2.2), $F_2'' \not\approx H$ if $a_{16}=0$. Then we can assume that

$$a_{16}=1. \tag{H2.115}$$

By $f'_3 \leftarrow f'_3 + b_{36}f'_4$, (H2.110), (H2.114) and (H2.115), we have that

$$F \approx H + x_7((x_1 + a_{15}x_5 + x_6)(x_4 + b_{25}x_5) + (x_3 + bx_5)(b_{25}x_5 + x_6)). \tag{H2.116}$$

By $x_7 \leftarrow x_4 + b_{25}x_5 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx H + (x_3 + bx_5)(x_4 + b_{25}x_5)(b_{25}x_5 + x_6) \\ &\approx (x_3 + \bar{b}x_5)(x_4 + b_{25}x_5)(b_{25}x_5 + x_6) + x_1x_2x_3 + x'_1x'_2x'_3. \end{aligned} \tag{H2.117}$$

If $b = 1$ or $\bar{b} = b_{25}$, then $F_{x_7=0} \not\approx H$. We can assume that

$$b = b_{25} = 0. \tag{H2.118}$$

By $x_7 \leftarrow x_6 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx H + (x_1 + a_{15}x_5)x_4x_6 \\ &\approx (x_1 + \bar{a}_{15}x_5)x_4x_6 + x_1x_2x_3 + x'_1x'_2x'_3. \end{aligned} \tag{H2.119}$$

If $a_{15} = 1$, then $F_{x_7=0} \not\approx H$. Then, we can assume that

$$a_{15} = 0, \tag{H2.120}$$

and that

$$F \approx H + x_7((x_1 + x_6)x_4 + x_3x_6). \tag{H2.121}$$

By $x_2 \leftrightarrow x_3$ and $x_5 \leftrightarrow x_6$, this is equivalent to (H2.85).

3) $h = 1$.

$$F \approx H + x_7f_1f_2, \tag{H2.122}$$

where $f_i = \sum_{j=1}^{6} a_{ij}x_j$ for $1 \le i \le 2$, and f_1 and f_2 are linearly independent.

For $1 \le i < j \le 6$, let A_{ijh} be the submatrix consisting of the i-th, j-th and h-th columns of $\begin{pmatrix} a_{11}, a_{12}, \cdots, a_{16} \\ a_{21}, a_{22}, \cdots, a_{26} \end{pmatrix}$.

3.1) The smallest rank of A_{123}, A_{456} and $A_{123} + A_{456}$ is less than two. Without loss of generality, assume that the rank of A_{123} is less than two, and $a_{11} = a_{12} = a_{13} = 0$ and $a_{14} = 1$. By $x_1 \leftarrow x_1 + a_{15}x_2 + a_{16}x_3$, $x_4 \leftarrow x_4 + a_{15}x_5 + a_{16}x_6$, $x_i \leftarrow x_i$ for $i = 2, 3, 5, 6$,

$$F \approx x_1x_2x_3 + x_4x_5x_6 + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6) + x_4x_7g, \tag{H2.123}$$

where $g = \sum_{j=1,j\neq 4}^{6} b_j x_j$.

3.1.1) $b_1 = b_2 = b_3 = 0$. Then b_5 or $b_6 = 1$. By $x_2 \leftarrow b_5 x_2 + b_6 x_3$, $x_5 \leftarrow b_5 x_5 + b_6 x_6$, $x_i \leftarrow x_i$ for $i = 1, 3, 4, 6$,

$$\begin{aligned} F &\approx H + x_4 x_5 x_7 \\ &\approx x_1 x_2 x_3 + x_4 x_5 (x_6 + x_7) + (x_1 + x_4)(x_2 + x_5)(x_3 + x_6). \end{aligned} \tag{H2.124}$$

Hence, $\tilde{\nu}(F) \leq 3$.

3.1.2) $b_1 = 1$, $b_2 = b_3 = 0$.

3.1.2.1) $b_5 = b_6 = 0$.

$$[7.4.1\text{-}2]: \quad F \approx H + x_1 x_4 x_7. \tag{H2.125}$$

3.1.2.2) Without loss of generality, assume that $b_5 = 1$. By $x_2 \leftarrow x_2 + b_6 x_3$, $x_5 \leftarrow x_5 + b_6 x_6$, $x_i \leftarrow x_i$ for $i = 1, 3, 4, 6$.

$$F \approx H + (x_1 + x_5) x_4 x_7. \tag{H2.126}$$

By $x_7 \leftarrow x_6 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx H + (x_1 + x_5) x_4 x_6 \\ &\approx x_1 (x_2 x_3 + x_4 x_6) + x_1' x_2' x_3' \\ &\not\approx H. \end{aligned} \tag{H2.127}$$

3.1.3) Without loss of generality, assume that $b_2 = 1$.

3.1.3.1) $b_5 = b_6 = 0$. By $x_2 \leftarrow b_1 x_1 + x_2 + b_3 x_3$, $x_5 \leftarrow b_1 x_4 + x_5 + b_3 x_6$, $x_i \leftarrow x_i$ for $i = 1, 3, 4, 6$,

$$F \approx H + x_2 x_4 x_7. \tag{H2.128}$$

By $x_7 \leftarrow x_6 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx H + x_2 x_4 x_6 \\ &\approx x_2' (x_1' x_3' + x_4 x_6) + x_1 x_2 x_3 \\ &\not\approx H. \end{aligned} \tag{H2.129}$$

3.1.3.2) $b_5 = 1$ and $b_3 = b_6$. By $x_2 \leftarrow b_1 x_1 + x_2 + b_3 x_3$, $x_5 \leftarrow b_1 x_4 + x_5 + b_6 x_6$, $x_i \leftarrow x_i$ for $i = 1, 3, 4, 6$.

$$F \approx H + (x_2 + x_5) x_4 x_7. \tag{H2.130}$$

By $x_1 \leftarrow x_1 + x_4$, $x_2 \leftarrow x_2 + x_5$, $x_3 \leftarrow x_3 + x_6$, $x_i \leftarrow x_i$ for $4 \le i \le 6$.

$$F \approx H + x_2x_4x_7. \tag{H2.131}$$

3.1.3.3) x_4, $b_1x_4 + x_5 + b_3x_6$ and $b_5x_5 + b_6x_6$ are linearly independent, that is, $b_3b_5 + b_6 = 1$. By $x_2 \leftarrow b_1x_1 + x_2 + b_3x_3$, $x_5 \leftarrow b_1x_4 + x_5 + b_3x_6$, $x_3 \leftarrow b_5x_2 + b_6x_3$, $x_6 \leftarrow b_5x_5 + b_6x_6$, $x_i \leftarrow x_i$ for $i = 1, 4$.

$$F \approx H + (x_2 + x_6)x_4x_7. \tag{H2.132}$$

By $x_7 \leftarrow x_5 + x_7$,

$$\begin{aligned} F_{x_7=0} &\approx H + (x_2 + x_6)x_4x_5, \\ &\approx x_2(x_1x_3 + x_4x_5) + x_1'x_2'x_3' \\ &\not\approx H. \end{aligned} \tag{H2.133}$$

3.2) The ranks of A_{123}, A_{456} and $A_{123} + A_{456}$ are all two. Let $\mathbf{a}_i = (a_{i1}, a_{i2}, a_{i3})$ and $\mathbf{b}_i = (a_{i4}, a_{i5}, a_{i6})$ for $i = 1, 2$. Then, $\mathbf{a}_i \neq \mathbf{0}$, $\mathbf{b}_i \neq \mathbf{0}$, $\mathbf{a}_i + \mathbf{b}_i \neq \mathbf{0}$, $\mathbf{a}_1 + \mathbf{a}_2 \neq \mathbf{0}$, $\mathbf{b}_1 + \mathbf{b}_2 \neq \mathbf{0}$, $\mathbf{a}_1 + \mathbf{b}_1 + \mathbf{a}_2 + \mathbf{b}_2 \neq \mathbf{0}$.

3.2.1) $\mathbf{b}_1$ and $\mathbf{b}_2$ are linear sums of $\mathbf{a}_1$ and $\mathbf{a}_2$. Without loss of generality, we can assume that

$$\begin{cases} \mathbf{b}_1 = \mathbf{a}_2, \\ \mathbf{b}_2 = \mathbf{a}_1 + \mathbf{a}_2. \end{cases} \tag{H2.134}$$

By $x_1 \leftarrow a_{11}x_1 + a_{12}x_2 + a_{13}x_3$, $x_4 \leftarrow a_{11}x_4 + a_{12}x_5 + a_{13}x_6$, $x_2 \leftarrow a_{21}x_1 + a_{22}x_2 + a_{23}x_3$, $x_5 \leftarrow a_{21}x_4 + a_{22}x_5 + a_{23}x_6$.

$$F \approx H + (x_1 + x_5)(x_2 + x_4 + x_5)x_7 \approx (H2.105). \tag{H2.135}$$

3.2.2) $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{b}_1$ are linearly independent. Then, $\mathbf{b}_2 = \alpha\mathbf{a}_1 + \beta\mathbf{a}_2 + \gamma\mathbf{b}_1$. If $\gamma = 1$, then take $f_1 \leftarrow f_1$ and $f_2 \leftarrow f_1 + f_2$. Hence we can assume that $\alpha = 1$ and $\gamma = 0$. By $x_1 \leftarrow a_{11}x_1 + a_{12}x_2 + a_{13}x_3$, $x_4 \leftarrow a_{11}x_4 + a_{12}x_5 + a_{13}x_6$, $x_2 \leftarrow a_{21}x_1 + a_{22}x_2 + a_{23}x_3$, $x_5 \leftarrow a_{21}x_4 + a_{22}x_5 + a_{23}x_6$. $x_3 \leftarrow a_{14}x_1 + a_{15}x_2 + a_{16}x_6$, $x_6 \leftarrow a_{14}x_4 + a_{15}x_5 + a_{16}x_6$.

$$F \approx H + (x_1 + x_6)(x_2 + x_4 + \beta x_5)x_7. \tag{H2.136}$$

By $x_7 \leftarrow x_6 + x_7$,

$$F_{x_7=0} \approx H + x_1(x_2 + x_4 + \beta x_5)x_6$$

$$
\begin{aligned}
&\approx x_1(x_2x_3 + x_2'x_3' + x_2x_6) + x_4(x_5x_6 + x_2'x_3') + x_1(x_4 + \beta x_5)x_6 \\
&\approx \begin{cases} x_1x_5x_3' + x_4((x_1 + x_5)x_6 + x_2'x_3') \not\approx H, & \text{for} \quad \beta = 0, \\ x_3'(x_1x_5 + x_2'x_4) + (x_1 + x_4)(x_4 + x_5)x_6 \not\approx H, & \text{for} \quad \beta = 1. \end{cases}
\end{aligned} \tag{H2.137}
$$