Equivalence of the Generalized Complementarity Problem to Differentiable Unconstrained Minimization

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Abstract. We consider an unconstrained minimization reformulation of the generalized complementarity problem (GCP). The merit function introduced here is differentiable and has the property that its global minimizers coincide with the solutions of GCP. Conditions for its stationary points to be global minimizers are given. Moreover, it is shown that the level sets of the merit function are bounded under suitable assumptions. We also show that the merit function provides global error bounds for GCP. These results are based on a condition which reduces to the condition of the uniform P-function when GCP is specialized to the nonlinear complementarity problem. This condition also turns out to be useful in proving the existence and uniqueness of a solution for GCP itself. Finally we obtain as a byproduct an error bound result with the natural residual for GCP.

Key words. Generalized complementarity problem, nonlinear complementarity problem, unconstrained minimization, stationary point, bounded level set, global error bound.

AMS (MOS) subject classifications. 90C33, 90C30.

Abbreviated title. Generalized Complementarity Problem.

1 Introduction

Let $F, G : \Re^n \to \Re^n$ be any two functions, and consider the *generalized complementarity problem*, which is to find a vector $x^* \in \Re^n$ satisfying the conditions

$$F(x) \geq 0, \quad G(x) \geq 0, \quad F(x)^T G(x) = 0.$$

We denote this problem by GCP(F,G).

The above formulation of GCP(F,G) covers some related problems studied in the literature; for example, if $G(x) \equiv x$, then GCP(F,G) reduces to the well–known nonlinear complementarity problem, NCP(F) for short. For affine functions F, i.e., $F(x) = Mx + q$ with a matrix $M \in \Re^{n\times n}$ and a vector $q \in \Re^n$, NCP(F) is called a linear complementarity problem and denoted by LCP(q, M). On the other hand, if we write $G(x) = x - H(x)$ with some function $H : \Re^n \to \Re^n$, then GCP(F,G) can be identified with the so–called implicit complementarity problem (sometimes also called quasi complementarity problem) as considered, e.g., in [25, 24, 10].

In a recent paper, Tseng, Yamashita and Fukushima [31] have studied the following unconstrained minimization formulation for GCP(F,G):

$$\min_{x\in\Re^n} \; M(x;\alpha) := \sum_{i\in I} \varphi_{MS}(F_i(x), G_i(x); \alpha),$$

where $\alpha > 1$ is any fixed parameter and

$$\varphi_{MS}(a,b;\alpha) := ab + \frac{1}{2\alpha}\left(\max{}^2\{0, a-\alpha b\} - a^2 + \max{}^2\{0, b - \alpha a\} - b^2\right)$$

is the basis of Mangasarian and Solodov's implicit Lagrangian [22]. (In [31], GCP(F,G) is dealt with under a more general setting and a modification of Fukushima's regularized gap function [8] is also discussed.) Mangasarian and Solodov [22] proved that $\varphi_{MS}(a,b;\alpha) \geq 0$ for all $a, b \in \Re$ and $\varphi_{MS}(a,b;\alpha) = 0$ if and only if $a \geq 0, b \geq 0, ab = 0$. Hence $M(x;\alpha)$ is nonnegative and its global minimizers coincide with the solutions of GCP(F,G). The function φ_{MS} has some further nice properties which attracted several authors, see, e.g., Luo, Mangasarian, Ren and Solodov [18], Yamashita and Fukushima [32], Kanzow [15] and Jiang [12].

The approach presented in this paper is similar, but we use a different merit function which is based on the function

$$\varphi_F(a,b) := \sqrt{a^2+b^2} - a - b. \tag{1}$$

This function is due to Fischer [4] and also enjoys the property that $\varphi_F(a,b) = 0$ if and only if $a \geq 0, b \geq 0, ab = 0$. Of course, φ_F could become negative, hence some authors propose to use the square of this function, namely

$$\varphi(a,b) := \frac{1}{2}\left(\sqrt{a^2+b^2} - a - b\right)^2. \tag{2}$$

The functions (1) and (2) are the basis of several recent papers on linear and nonlinear complementarity problems as well as on constrained optimization problems, see, e.g., Fischer [5, 6, 7], Tseng [30], Jiang [12], Qi and Jiang [13, 29], Facchinei and Soares [2, 3], Geiger and Kanzow [9] and Kanzow [14, 15, 16, 17]. Here we use the function (2) in order to reformulate GCP(F, G) as the unconstrained minimization problem

$$\min_{x \in \Re^n} \quad \Psi(x) := \sum_{i \in I} \varphi(F_i(x), G_i(x)). \tag{3}$$

Like the function $M(x; \alpha)$, there is a one–to–one correspondence between global minimizers of Ψ and solutions of GCP(F, G).

The paper is organized as follows. In Section 2, we recall some definitions and basic facts which are needed in the subsequent analysis. Section 3 is devoted to examine sufficient conditions for any stationary point of the merit function Ψ to be a solution of GCP(F, G). In Section 4, we prove that the Hessian matrix of Ψ is positive definite at a solution of GCP(F, G) under suitable assumptions. In Section 5, a new result on the existence and uniqueness of a solution to GCP(F, G) is given. This result relies on a condition (see (10)), which reduces to the condition of the uniform P-function in the case of the nonlinear complementarity problem. This condition also turns out to be important in Sections 6 and 7. In Section 6, we give some sufficient conditions for the level sets of the merit function Ψ to be bounded, whereas in Section 7 we prove that the function Ψ provides global error bounds for GCP(F, G) under appropriate conditions. Moreover we obtain as a byproduct an error bound result with the natural residual for GCP(F, G). We conclude with some final remarks in Section 8.

Notation: Throughout this paper, the index set $\{1, \ldots, n\}$ is abbreviated by the upper case letter I. If F is a (continuously differentiable) vector–valued function, its ith component function is denoted by F_i. For each i, the gradient of F_i at x, denoted by $\nabla F_i(x)$, is a column vector whose jth component is $\frac{\partial F_i}{\partial x_j}(x)$. The Jacobian of F at x, denoted by $F'(x)$, is a matrix whose ith row is $\nabla F_i(x)^T$, where T denotes transposition. Finally, $\|x\|$ denotes the Euclidean norm of a vector $x \in \Re^n$.

2 Mathematical Background

We first recall the definitions of some classes of functions.

2.1 Definition. A mapping $F : \Re^n \to \Re^n$ is called a
(a) *P_0-function* $\iff \forall x, y \in \Re^n, x \neq y, \exists i \in I : x_i \neq y_i$ and

$$[x_i - y_i]\,[F_i(x) - F_i(y)] \geq 0,$$

(b) *P-function* $\iff \forall x, y \in \Re^n, x \neq y :$

$$\max_{i \in I}\,[x_i - y_i]\,[F_i(x) - F_i(y)] > 0,$$

(c) *uniform P-function* (with modulus $\mu > 0$) $\iff \forall x, y \in \Re^n :$

$$\max_{i \in I}\,[x_i - y_i]\,[F_i(x) - F_i(y)] \geq \mu \,\|x - y\|^2.$$

$\square$

Let F be an affine function given by $F(x) = Mx + q$, where $M \in \Re^{n \times n}$ and $q \in \Re^n$. Then M is called a *P_0-matrix* if F is a P_0-function, and M is said to be a *P-matrix* if

F is a P-function. (In the affine case, F is a P-function if and only if it is a uniform P-function.) Note that $M \in \Re^{n\times n}$ is a P_0-matrix if and only if, for all $x \in \Re^n, x \neq 0$, there exists an index $i \in I$ such that $x_i \neq 0$ and $x_i[Mx]_i \geq 0$.

For nonlinear functions F, the following correspondence between P_0-functions and P_0-matrices is due to Moré and Rheinboldt [23, Corollary 5.3 and Theorem 5.8].

2.2 LEMMA. *A continuously differentiable function $F : \Re^n \to \Re^n$ is a P_0-function if and only if the Jacobian matrices $F'(x)$ are P_0-matrices for all $x \in \Re^n$.* □

The following classical result (see, e.g., Harker and Pang [11, Theorem 3.9]) will also be used in our analysis.

2.3 LEMMA. *Assume that $F : \Re^n \to \Re^n$ is a continuous uniform P-function. Then NCP(F) has a unique solution.* □

For the linear complementarity problem LCP(q, M), Lemma 2.3 can be sharpened as follows: LCP(q, M) has a unique solution for all $q \in \Re^n$ if and only if F is a (uniform) P-function, i.e., if M is a P-matrix, see, e.g., Cottle, Pang and Stone [1, Theorem 3.3.7].

3 Stationary Points

Some elementary, but important properties of the function φ as defined in (2) are summarized in the following result.

3.1 LEMMA. *The function φ defined by (2) satisfies the following properties:*
(a) φ is continuously differentiable on $\Re^2$, in particular $\nabla\varphi(0,0) = (0,0)$.
(b) $\varphi(a,b) \geq 0$ for all $(a,b) \in \Re^2$.
(c) $\varphi(a,b) = 0 \iff a \geq 0, b \geq 0, ab = 0$.
(d) $\frac{\partial\varphi}{\partial a}(a,b)\frac{\partial\varphi}{\partial b}(a,b) \geq 0$ for all $(a,b) \in \Re^2$.
(e) $\frac{\partial\varphi}{\partial a}(a,b)\frac{\partial\varphi}{\partial b}(a,b) = 0 \iff \varphi(a,b) = 0$.

PROOF. See Kanzow [17, Lemma 2.1]. □

As already noted in the introduction, Lemma 3.1 and the definition of Ψ (cf. (3)) yield the following characterization of solutions of GCP(F, G).

3.2 THEOREM. *If GCP(F, G) has a nonempty solution set, then $x^* \in \Re^n$ is a solution of GCP(F, G) if and only if $\Psi(x^*) = 0$, i.e., x^* is a global minimum of the nonnegative function Ψ.*

Note that the function Ψ may have a global minimum $x^* \in \Re^n$ with $\Psi(x^*) > 0$ when problem GCP(F, G) has no solutions.

The subject of the remaining part of this section is to find suitable conditions which guarantee that every stationary point of Ψ is a global minimum of this function.

3.3 THEOREM. *Let $F, G : \Re^n \to \Re^n$ be continuously differentiable. Let $x^* \in \Re^n$ be a stationary point of Ψ such that $G'(x^*)$ is nonsingular and the product $F'(x^*)G'(x^*)^{-1}$ is a P_0-matrix. Then x^* is a solution of GCP(F, G).*

PROOF. Assume that $x^* \in \Re^n$ is a stationary point of Ψ, i.e.

$$\nabla\Psi(x^*) = \sum_{i\in I} \frac{\partial\varphi}{\partial a}(F_i(x^*), G_i(x^*))\nabla F_i(x^*) + \sum_{i\in I} \frac{\partial\varphi}{\partial b}(F_i(x^*), G_i(x^*))\nabla G_i(x^*) = 0.$$

This equation can be rewritten as

$$F'(x^*)^T \frac{\partial\varphi}{\partial a}(F(x^*), G(x^*)) + G'(x^*)^T \frac{\partial\varphi}{\partial b}(F(x^*), G(x^*)) = 0, \tag{4}$$

where

$$\frac{\partial\varphi}{\partial a}(F(x^*), G(x^*)) := \left(\frac{\partial\varphi}{\partial a}(F_1(x^*), G_1(x^*)), \ldots, \frac{\partial\varphi}{\partial a}(F_n(x^*), G_n(x^*))\right)^T \in \Re^n$$

and similarly

$$\frac{\partial\varphi}{\partial b}(F(x^*), G(x^*)) := \left(\frac{\partial\varphi}{\partial b}(F_1(x^*), G_1(x^*)), \ldots, \frac{\partial\varphi}{\partial b}(F_n(x^*), G_n(x^*))\right)^T \in \Re^n.$$

Since $G'(x^*)$ is nonsingular, we can premultiply equation (4) by $(G'(x^*)^T)^{-1}$ to obtain

$$(F'(x^*)G'(x^*)^{-1})^T \frac{\partial\varphi}{\partial a}(F(x^*), G(x^*)) + \frac{\partial\varphi}{\partial b}(F(x^*), G(x^*)) = 0. \tag{5}$$

For simplicity of notation, define $M := (F'(x^*)G'(x^*)^{-1})^T$. Then, (5) can be rewritten componentwise as follows:

$$\left[M\frac{\partial\varphi}{\partial a}(F(x^*), G(x^*))\right]_i + \frac{\partial\varphi}{\partial b}(F_i(x^*), G_i(x^*)) = 0 \quad (i \in I).$$

Premultiplying this equation by $\frac{\partial\varphi}{\partial a}(F_i(x^*), G_i(x^*))$ yields

$$\frac{\partial\varphi}{\partial a}(F_i(x^*), G_i(x^*)) \left[M\frac{\partial\varphi}{\partial a}(F(x^*), G(x^*))\right]_i + \frac{\partial\varphi}{\partial a}(F_i(x^*), G_i(x^*))\frac{\partial\varphi}{\partial b}(F_i(x^*), G_i(x^*)) = 0 \tag{6}$$

for all $i \in I$. Assume that $\frac{\partial\varphi}{\partial a}(F(x^*), G(x^*)) \neq 0$. Since M is a P_0-matrix, there exists an index $i_0 \in I$ with $\frac{\partial\varphi}{\partial a}(F_{i_0}(x^*), G_{i_0}(x^*)) \neq 0$ and

$$\frac{\partial\varphi}{\partial a}(F_{i_0}(x^*), G_{i_0}(x^*)) \left[M\frac{\partial\varphi}{\partial a}(F(x^*), G(x^*))\right]_{i_0} \geq 0. \tag{7}$$

Because of Lemma 3.1 (d), we also have

$$\frac{\partial\varphi}{\partial a}(F_{i_0}(x^*), G_{i_0}(x^*))\frac{\partial\varphi}{\partial b}(F_{i_0}(x^*), G_{i_0}(x^*)) \geq 0. \tag{8}$$

From (6), (7), (8) and Lemma 3.1 (e), we therefore have

$$\varphi(F_{i_0}(x^*), G_{i_0}(x^*)) = 0,$$

i.e., $(F_{i_0}(x^*), G_{i_0}(x^*))$ is a global minimizer of the nonnegative function φ. Since φ is differentiable, we have in particular $\frac{\partial\varphi}{\partial a}(F_{i_0}(x^*), G_{i_0}(x^*)) = 0$, a contradiction to the choice of the index i_0. Hence $\frac{\partial\varphi}{\partial a}(F(x^*), G(x^*)) = 0$, which, by Lemma 3.1 (e), implies $\varphi(F_i(x^*), G_i(x^*)) = 0$ for all $i \in I$. Consequently, x^* is a solution of GCP(F,G) by Theorem 3.2. $\square$

We note that the proof of Theorem 3.3 also shows that GCP(F,G) has a nonempty solution set if and only if the merit function Ψ has a stationary point.

If $G(x) \equiv x$, so that GCP(F,G) reduces to the nonlinear complementarity problem NCP(F), then Theorem 3.3 says that every stationary point of Ψ is a solution of NCP(F) if $F'(x^*)$ is a P_0-matrix. By Lemma 2.2, this is true if F is a continuously differentiable P_0-function, i.e., we reobtain a recent result of Facchinei and Soares [2, Theorem 4.1].

We also note that Tseng, Yamashita and Fukushima [31, Theorem 4.2] proved a similar result for Mangasarian–Solodov's implicit Lagrangian $M(x;\alpha)$, but they had to assume that the matrix $F'(x^*)G'(x^*)^{-1}$ is positive definite. So our assumption is weaker; in particular, it is satisfied if $F'(x^*)G'(x^*)^{-1}$ is a positive semidefinite matrix.

4 Isolated Minima

Here we prove that, under certain assumptions, the Hessian matrix $\nabla^2\Psi(x)$ exists and is positive definite. The proof of this result is modelled after the corresponding proof for NCP(F) given in [15]. We first restate some easily established properties of the function φ.

4.1 Lemma. *Let $(a,b) \in \Re^2$ be any vector such that $a,b \geq 0, ab = 0$ and $a+b > 0$. Then φ is twice continuously differentiable at (a,b) and*

$$\begin{aligned}
\frac{\partial\varphi}{\partial a}(a,b) &= 0, \\
\frac{\partial\varphi}{\partial b}(a,b) &= 0, \\
\frac{\partial^2\varphi}{\partial a\partial b}(a,b) &= 0, \\
\frac{\partial^2\varphi}{\partial a^2}(a,b) &= 0 \quad \text{if} \quad a > 0 = b, \\
\frac{\partial^2\varphi}{\partial a^2}(a,b) &\neq 0 \quad \text{if} \quad b > 0 = a, \\
\frac{\partial^2\varphi}{\partial b^2}(a,b) &\neq 0 \quad \text{if} \quad a > 0 = b, \\
\frac{\partial^2\varphi}{\partial b^2}(a,b) &= 0 \quad \text{if} \quad b > 0 = a.
\end{aligned}$$

Proof. The assertions can be easily verified; see Kanzow [15]. $\square$

For an arbitrary solution $x^* \in \Re^n$ of GCP(F,G), let us introduce the index sets

$$\begin{aligned}
I_F^* &:= \{i \in I \,|\, F_i(x^*) = 0\}, \\
I_G^* &:= \{i \in I \,|\, G_i(x^*) = 0\}.
\end{aligned}$$

A solution x^* of GCP(F,G) is called a *strict complementary solution* (or *nondegenerate solution*) if $I_F^* \cap I_G^* = \emptyset$.

4.2 THEOREM. *Assume that $F, G : \Re^n \to \Re^n$ are twice continuously differentiable. Let $x^* \in \Re^n$ be a strict complementary solution of GCP(F,G), and suppose that the gradients $\nabla F_i(x^*)$ $(i \in I_F^*)$ and $\nabla G_i(x^*)$ $(i \in I_G^*)$ are linearly independent. Then the Hessian matrix $\nabla^2\Psi(x^*)$ exists and is nonsingular.*

PROOF. Since x^* is a strict complementary solution of GCP(F,G), it follows from Lemma 4.1 that the function Ψ is twice continuously differentiable at x^*. Let $\Psi(x) = \sum_{i\in I}\Psi_i(x)$, where

$$\Psi_i(x) := \varphi(F_i(x), G_i(x)) \quad (i \in I).$$

We have

$$\nabla\Psi_i(x) = \frac{\partial\varphi}{\partial a}(F_i(x), G_i(x))\nabla F_i(x) + \frac{\partial\varphi}{\partial b}(F_i(x), G_i(x))\nabla G_i(x)$$

and

$$\begin{aligned}\nabla^2\Psi_i(x) \;=\;& \frac{\partial^2\varphi}{\partial a^2}(F_i(x), G_i(x))\nabla F_i(x)\nabla F_i(x)^T \\ &+ \frac{\partial^2\varphi}{\partial a\partial b}(F_i(x), G_i(x))\nabla F_i(x)\nabla G_i(x)^T + \frac{\partial\varphi}{\partial a}(F_i(x), G_i(x))\nabla^2 F_i(x) \\ &+ \frac{\partial^2\varphi}{\partial b^2}(F_i(x), G_i(x))\nabla G_i(x)\nabla G_i(x)^T \\ &+ \frac{\partial^2\varphi}{\partial a\partial b}(F_i(x), G_i(x))\nabla G_i(x)\nabla F_i(x)^T + \frac{\partial\varphi}{\partial b}(F_i(x), G_i(x))\nabla^2 G_i(x).\end{aligned}$$

From Lemma 4.1, we therefore obtain at $x = x^*$

$$\begin{aligned}\nabla^2\Psi(x^*) &= \sum_{i\in I}\nabla^2\Psi_i(x^*) \\ &= \sum_{i\in I_F^*}\nabla^2\Psi_i(x^*) + \sum_{i\in I_G^*}\nabla^2\Psi_i(x^*) \\ &= \sum_{i\in I_F^*}\frac{\partial^2\varphi}{\partial a^2}(F_i(x^*), G_i(x^*))\nabla F_i(x^*)\nabla F_i(x^*)^T \\ &\quad + \sum_{i\in I_G^*}\frac{\partial^2\varphi}{\partial a^2}(F_i(x^*), G_i(x^*))\nabla G_i(x^*)\nabla G_i(x^*)^T.\end{aligned}$$

For simplicity, we denote $\frac{\partial^2\varphi}{\partial a^2}(F_i(x^*), G_i(x^*))$ and $\frac{\partial^2\varphi}{\partial a^2}(F_i(x^*), G_i(x^*))$ by $\frac{\partial^2\varphi}{\partial a^2}$ and $\frac{\partial^2\varphi}{\partial a^2}$, respectively. Let $p \in \Re^n$ be an arbitrary vector such that $\nabla^2\Psi(x^*)p = 0$, i.e.,

$$\sum_{i\in I_F^*}\frac{\partial^2\varphi}{\partial a^2}\nabla F_i(x^*)^T p\nabla F_i(x^*) + \sum_{i\in I_G^*}\frac{\partial^2\varphi}{\partial b^2}\nabla G_i(x^*)^T p\nabla G_i(x^*) = 0. \tag{9}$$

Since the gradients $\nabla F_i(x^*)$ $(i \in I_F^*)$ and $\nabla G_i(x^*)$ $(i \in I_G^*)$ are linearly independent, we obtain from (9)

$$\begin{aligned}\frac{\partial^2\varphi}{\partial a^2}\nabla F_i(x^*)^T p &= 0 \quad (i \in I_F^*), \\ \frac{\partial^2\varphi}{\partial a^2}\nabla G_i(x^*)^T p &= 0 \quad (i \in I_G^*).\end{aligned}$$

Again using Lemma 4.1, we therefore have

$$\begin{aligned} \nabla F_i(x^*)^T p &= 0 \quad (i \in I_F^*), \\ \nabla G_i(x^*)^T p &= 0 \quad (i \in I_G^*). \end{aligned}$$

From the linear independence of the gradients $\nabla F_i(x^*)$ $(i \in I_F^*)$ and $\nabla G_i(x^*)$ $(i \in I_G^*)$, we finally get $p = 0$. Consequently, the Hessian matrix $\nabla^2 \Psi(x^*)$ is indeed nonsingular. □

4.3 COROLLARY. *Under the assumptions of Theorem 4.2, the Hessian matrix* $\nabla^2 \Psi(x^*)$ *is positive definite.*

PROOF. Since x^* is a solution of GCP(F, G), it is a global minimizer of the objective function Ψ (see Theorem 3.2). So it follows from the second-order necessary optimality condition for unconstrained minimization that the Hessian matrix $\nabla^2 \Psi(x^*)$ is at least positive semidefinite. Since this matrix is symmetric and since it is nonsingular by Theorem 4.2, it must be positive definite. □

It is not difficult to see that the function φ_{MS} (the basis of Mangasarian and Solodov's implicit Lagrangian) also satisfies the conditions stated in Lemma 4.1. Hence Theorem 4.2 and Corollary 4.3 remain true for the implicit Lagrangian, i.e., we reobtain Theorem 5.1 of Tseng, Yamashita and Fukushima [31] in this way.

We also note that it is not necessary in Theorem 4.2 (and Corollary 4.3) to assume explicitly that x^* is a nondegenerate solution of GCP(F, G), since the linear independence assumption on the gradients of the active constraints imply that x^* is nondegenerate (since there cannot exist more than n linearly independent gradients in $\Re^n$).

The assertion of Corollary 4.3 is important since it guarantees that the solution x^* is an isolated solution of GCP(F, G), which in turn implies that any method for solving the unconstrained optimization problem

$$\min_{x \in \Re^n} \Psi(x)$$

achieves its known (fast) rate of convergence in a neighbourhood of such a solution.

5 Existence and Uniqueness of Solutions

In this section, we first prove that GCP(F, G) has at most one solution if the functions F and G satisfy the following condition: There exists a constant $\rho > 0$ such that

$$\max_{i \in I} [F_i(x) - F_i(y)] [G_i(x) - G_i(y)] \geq \rho \|x - y\|^2 \quad \text{for all} \quad x, y \in \Re^n. \tag{10}$$

This condition also plays a central role in our existence result, Theorem 5.3 below, as well as in Sections 6 and 7. In the case of the nonlinear complementarity problem, where $G(x) \equiv x$, condition (10) reduces to the definition of F being a uniform P-function, which is known to be a sufficient condition for NCP(F) to have a unique solution (see Lemma 2.3).

5.1 LEMMA. *Assume that* $F, G : \Re^n \to \Re^n$ *satisfy condition (10) for some* $\rho > 0$. *Then GCP(F, G) has at most one solution.*

PROOF. Let $x^*, y^* \in \Re^n$ be two solutions of GCP(F,G). According to (10), there exists an index $i \in I$ such that

$$\begin{aligned} 0 &\le \rho\|x^* - y^*\|^2 \\ &\le [F_i(x^*) - F_i(y^*)]\,[G_i(x^*) - G_i(y^*)] \\ &= -F_i(y^*)G_i(x^*) - F_i(x^*)G_i(y^*) \\ &\le 0, \end{aligned}$$

since x^*, y^* are solutions of GCP(F,G). Hence we have $x^* = y^*$. □

In order to guarantee the existence of a solution of GCP(F,G), we need an additional assumption which is fairly strong.

5.2 LEMMA. *Assume that $F, G : \Re^n \to \Re^n$ satisfy condition (10) for some $\rho > 0$, and let G be a Lipschitz–continuous one–to–one mapping. Then the mapping $H : \Re^n \to \Re^n$ defined by $H := F \circ G^{-1}$ is a uniform P-function.*

PROOF. For all $x, y \in \Re^n$, there exist unique $u, v \in \Re^n$ with $x = G^{-1}(u)$ and $y = G^{-1}(v)$, since G is one–to–one. By assumption, there is a Lipschitz–constant $L > 0$ such that

$$\|G(x) - G(y)\| \le L\|x - y\| \qquad \text{for all} \quad x, y \in \Re^n.$$

From this we obtain

$$\|u - v\| \le L\|G^{-1}(u) - G^{-1}(v)\| \qquad \text{for all} \quad u, v \in \Re^n. \tag{11}$$

Using (10), (11) and the definition of H, we get for all $u, v \in \Re^n$

$$\begin{aligned} \max_{i\in I}\,[H_i(u) - H_i(v)](u_i - v_i) &= \max_{i\in I}\,[H_i(G(x)) - H_i(G(y))]\,[G_i(x) - G_i(y)] \\ &= \max_{i\in I}\,[F_i(x) - F_i(y)]\,[G_i(x) - G_i(y)] \\ &\ge \rho\|x - y\|^2 \\ &= \rho\|G^{-1}(u) - G^{-1}(v)\|^2 \\ &\ge \frac{\rho}{L^2}\|u - v\|^2. \end{aligned}$$

Therefore, H is a uniform P-function (with modulus $\mu := \rho/L^2$). □

5.3 THEOREM. *Assume that $F, G : \Re^n \to \Re^n$ satisfy condition (10) for some $\rho > 0, F$ is continuous and G is a Lipschitz–continuous homeomorphism. Then GCP(F,G) has a unique solution.*

PROOF. According to Lemma 5.2, the mapping $H : \Re^n \to \Re^n$ defined by $H := F \circ G^{-1}$ is a uniform P-function. Moreover, since F and G^{-1} are continuous, H is also a continuous function. So it follows from Lemma 2.3 that NCP(H), which is to find $u \in \Re^n$ such that

$$H(u) \ge 0, \quad u \ge 0, \quad u^T H(u) = 0, \tag{12}$$

has a unique solution $u^* \in \Re^n$. Since NCP(H) is easily shown to be equivalent to GCP(F,G), we can conclude that GCP(F,G) also has a unique solution $x^* = G^{-1}(u^*) \in \Re^n$. □

6 Bounded Level Sets

The proof of the first result in this section is based on a recent result of Facchinei and Soares [2] and Jiang and Qi [13] for nonlinear complementarity problems.

6.1 THEOREM. *Assume that $F : \Re^n \to \Re^n$ is continuous and $G : \Re^n \to \Re^n$ is a homeomorphism such that the mapping $H : \Re^n \to \Re^n$ defined by $H := F \circ G^{-1}$ is a uniform P-function. Then the level sets $\mathcal{L}_\Psi(x^0) := \{x \in \Re^n \mid \Psi(x) \leq \Psi(x^0)\}$ are compact for all $x^0 \in \Re^n$.*

PROOF. Let $x^0 \in \Re^n$ be arbitrary. Since F and G are continuous, it suffices to prove the boundedness of $\mathcal{L}_\Psi(x^0)$. Let $u = G(x)$, i.e., $x = G^{-1}(u)$. Then GCP(F, G) is equivalent to NCP(H); see (12). Recall that our merit function for GCP(F, G) is

$$\Psi(x) = \sum_{i \in I} \varphi(F_i(x), G_i(x)),$$

and denote the corresponding merit function for NCP(H) by Φ, i.e.,

$$\Phi(u) = \sum_{i \in I} \varphi(H_i(u), u_i).$$

These two merit functions are related by

$$\Psi(x) = \Phi(G(x))$$

and

$$\Phi(u) = \Psi(G^{-1}(u)).$$

Denote the level set of Φ by

$$\mathcal{L}_\Phi(u^0) := \{u \in \Re^n \mid \Phi(u) \leq \Phi(u^0)\},$$

where $u^0 = G(x^0)$. Since G is a homeomorphism, we have

$$\mathcal{L}_\Phi(u^0) = G\left(\mathcal{L}_\Psi(x^0)\right)$$

and

$$\mathcal{L}_\Psi(x^0) = G^{-1}\left(\mathcal{L}_\Phi(u^0)\right).$$

Since G and G^{-1} are continuous, this implies that $\mathcal{L}_\Phi(u^0)$ is compact if and only if $\mathcal{L}_\Psi(x^0)$ is compact. Since H is a continuous and uniform P-function, it follows from Theorem 4.2 in Facchinei and Soares [2] or Proposition 4 in Jiang and Qi [13] that $\mathcal{L}_\Phi(u^0)$ is bounded. Therefore $\mathcal{L}_\Psi(x^0)$ is also bounded. □

6.2 COROLLARY. *Assume that $F, G : \Re^n \to \Re^n$ satisfy the conditions of Theorem 5.3. Then the level sets $\mathcal{L}_\Psi(x^0)$ are bounded for all $x^0 \in \Re^n$.*

PROOF. This follows immediately from Lemma 5.2, Theorem 5.3 and Theorem 6.1, since the assumptions imply that $H := F \circ G^{-1}$ is a continuous and uniform P-function. □

We note that the function $\Phi(u)$ as introduced in the proof of Theorem 6.1 is also a possible merit function for GCP(F,G) (if $H = F \circ G^{-1}$ exists). However, this merit function needs the mapping G^{-1} explicitly, which is not very nice from a computational point of view. We now give a result on bounded level sets without using G^{-1} and without assuming the existence of G^{-1}. This result is based on the following simple lemma.

6.3 LEMMA. *If $\{a^k\}, \{b^k\} \subseteq \Re$ are two sequences with $|a^k| \to \infty$ and $|b^k| \to \infty$, then $\varphi(a^k, b^k) \to \infty$.*

PROOF. This follows immediately from Lemma 2.5 in [16]. □

6.4 THEOREM. *Assume that $F, G : \Re^n \to \Re^n$ are Lipschitz–continuous functions satisfying condition (10) for some $\rho > 0$. Then the level sets $\mathcal{L}_\Psi(x^0)$ are bounded for all $x^0 \in \Re^n$.*

PROOF. Let $x^0 \in \Re^n$ be fixed arbitrarily. Assume there exists an unbounded sequence $\{x^k\} \subseteq \mathcal{L}_\Psi(x^0)$. Then, by condition (10), we have

$$\begin{aligned} \rho \, \|x^k\|^2 &\le \max_{i \in I} \, [F_i(x^k) - F_i(0)] \, [G_i(x^k) - G_i(0)] \\ &\le [F_{i_0}(x^k) - F_{i_0}(0)] \, [G_{i_0}(x^k) - G_{i_0}(0)] \end{aligned} \tag{13}$$

for some index $i_0 \in I$ independent of k and a suitable subsequence of $\{x^k\}$. It is clear from (13) that at least one of the two sequences $\{F_{i_0}(x^k)\}$ and $\{G_{i_0}(x^k)\}$ is unbounded on this subsequence. We prove that both sequences are actually unbounded. Suppose this is not true. Without loss of generality, let $\{F_{i_0}(x^k)\}$ be unbounded and $\{G_{i_0}(x^k)\}$ be bounded. Then there exists a constant $c > 0$ such that

$$\begin{aligned} \rho \, \|x^k\|^2 &\le [F_{i_0}(x^k) - F_{i_0}(0)] \, [G_{i_0}(x^k) - G_{i_0}(0)] \\ &\le c \, [F_{i_0}(x^k) - F_{i_0}(0)] \\ &\le Lc \, \|x^k\|, \end{aligned} \tag{14}$$

where the first inequality comes from (13) and $L > 0$ denotes the Lipschitz constant of F. However, (14) contradicts the fact that $\{x^k\}$ is unbounded. Therefore $F_{i_0}(x^k)$ and $G_{i_0}(x^k)$ are both unbounded for the same index $i_0 \in I$. Because of Lemma 6.3 and Lemma 3.1 (b), this implies that $\{\Psi(x^k)\}$ is unbounded, which contradicts the assumption $\{x^k\} \subseteq \mathcal{L}_\Psi(x^0)$. This proves the theorem. □

It follows directly from the above proof that Theorem 6.4 still holds if condition (10) is replaced by the weaker condition

$$\max_{i \in I} \, [F_i(x) - F_i(0)] \, [G_i(x) - G_i(0)] \ge \rho \, \|x\|^2 \tag{15}$$

for all $x \in \Re^n$ and some $\rho > 0$ (actually, 0 can be replaced by any fixed vector in $\Re^n$). For the special case of a nonlinear complementarity problem, this condition is satisfied, e.g., for functions F which are strongly monotone, strongly copositive or uniform P-functions (on the entire space $\Re^n$). For GCP(F,G), however, condition (15) may not hold in general even if F and G are both strongly monotone individually, as shown by the following example.

6.5 Example. Let $n = 2$ and $F, G : \Re^2 \to \Re^2$ be defined by $F(x) := Mx$ and $G(x) := Nx$, where

$$M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and

$$N = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}.$$

The matrices M and N are symmetric and positive definite, so the functions F and G are strongly monotone. Consider the ray $(x_1, x_2) := (t, 2t)$ for $t \to \infty$. It is not difficult to see that $\Psi(x) = 0$ on this ray. Hence the level set $\mathcal{L}_\Psi(x^0)$ is not bounded for any $x^0 \in \Re^n$. This means that condition (15) is not satisfied since all other assumptions of Theorem 6.4 are valid. □

We stress that the only property of the function φ used in the proof of Theorem 6.4 is the one stated in Lemma 6.3. In particular, Theorem 6.4 remains true for Mangasarian–Solodov's implicit Lagrangian $M(x;\alpha)$ (see Section 1), if we are able to prove that the function φ_{MS} enjoys a property similar to the one stated in Lemma 6.3. In fact, we can show the following lemma, which implies that Theorem 6.4 is also true for Mangasarian–Solodov's implicit Lagrangian $M(x;\alpha)$.

6.6 Lemma. *Let $\alpha > 1$. If $\{a^k\}, \{b^k\} \subseteq \Re$ are two sequences with $|a^k| \to \infty$ and $|b^k| \to \infty$, then $\varphi_{MS}(a^k, b^k; \alpha) \to \infty$.*

Proof. Luo, Mangasarian, Ren and Solodov [18] showed in the proof of their Theorem 4.1 that there exist constants $c_1 > 0$ and $c_2 > 0$ such that the inequalities

$$c_1 \min{}^2\{a, b\} \le \varphi_{MS}(a, b; \alpha) \le c_2 \min{}^2\{a, b\} \tag{16}$$

hold for all $(a, b)^T \in \Re^2$. The assertion now follows immediately (only the left inequality of (16) is needed). □

6.7 Corollary. *Let $\alpha > 1$. Assume that $F, G : \Re^n \to \Re^n$ are Lipschitz–continuous functions satisfying condition (10) for some $\rho > 0$. Then the level sets $\mathcal{L}_M(x^0) := \{x \in \Re^n \,|\, M(x;\alpha) \le M(x^0;\alpha)\}$ are bounded for all $x^0 \in \Re^n$.* □

When specialized to the nonlinear complementarity problem NCP(F), Corollary 6.7 states that the level sets are bounded for a Lipschitz–continuous uniform P-function. Note that this result improves on a recent result of Yamashita and Fukushima [32, Theorem 2.3], where F was assumed to be strongly monotone and Lipschitz–continuous. On the other hand, Jiang [12, Proposition 9] very recently proves that the level sets are bounded for continuous uniform P-functions (no Lipschitz-condition is needed). His proof makes use of a lemma similar to our Lemma 6.6. His proof of this lemma, however, is longer since he did not exploit a condition like (16). The condition (16), on the other hand, will also be crucial in the following section.

7 Global Error Bounds

The analysis of this section is an extension of the one given by Mangasarian and Ren [21] for the nonlinear complementarity problem. We first restate the following definition of error bounds.

7.1 DEFINITION. Let $e : \Re^n \to \Re, X \subseteq \Re^n$ and let $dist(x, X) := \inf_{y \in X} \|x - y\|$ denote the distance of an arbitray vector x to the set X. Assume that GCP(F, G) has a nonempty solution set X^*.

(1) The function e is called a *residual* of GCP(F, G) if $e(x) \geq 0$ for all $x \in \Re^n$ and $e(x) = 0$ if and only if x solves GCP(F, G).

(2) A residual e is a *lower global error bound* for GCP(F, G) if there exists some constant $\tau_1 > 0$ such that

$$\tau_1 e(x) \leq dist(x, X^*) \qquad \text{for each} \quad x \in \Re^n.$$

(3) A residual e is an *upper global error bound* for GCP(F, G) if there exists some constant $\tau_2 > 0$ such that

$$dist(x, X^*) \leq \tau_2 e(x) \qquad \text{for each} \quad x \in \Re^n.$$

□

From the basic properties of the functions φ and φ_{MS}, it follows immediately that our merit function Ψ as well as Mangasarian–Solodov's implicit Lagrangian M with $\alpha > 1$ are residuals for GCP(F, G). The principal aim of this section is to prove that (the square roots of) these two functions provide lower and upper global error bounds for GCP(F, G).

For each $i \in I$, define

$$r_i(x) := \min\{F_i(x), G_i(x)\}.$$

Let $r(x)$ denote the vector with components $r_i(x)$ $(i \in I)$. It is clear that $\|r(x)\|$ provides a residual of GCP(F, G), and it is called the *natural* residual for GCP(F, G). In the special cases of LCP(q, M) and NCP(F), the natural residual has played an important role in establishing global error bounds, see, e.g., Pang [27], Mangasarian and Ren [20, 21] and Luo and Tseng [19]. It will play a similar role in our analysis for GCP(F, G).

We first prove a few technical results.

7.2 LEMMA. *For all $a, b, c, d \in \Re$, we have*

$$|\min\{a, b\} - \min\{c, d\}| \leq \max\{|a - c|, |b - d|\}.$$

PROOF. It suffices to consider the two cases (i) $a \leq b$ and $c \leq d$ and (ii) $a \leq b$ and $c \geq d$. In case (i), the inequality in the lemma trivially holds. In case (ii), we have

$$|\min\{a, b\} - \min\{c, d\}| = |a - d|.$$

This case can be further divided into the following two subcases: (ii-a) $a \le d$, (ii-b) $a > d$. In subcase (ii-a), we have $a \le d \le c$ and therefore $|a-d| \le |a-c|$, whereas in subcase (ii-b), we have $b \ge a > d$, which implies $|a-d| \le |b-d|$. From this the desired inequality follows immediately. □

7.3 LEMMA. *Assume that $F, G : \Re^n \to \Re^n$ are Lipschitz–continuous with (common) constant $L > 0$. Then, for all $i \in I$, we have*

$$|r_i(x)| \le L\|x - x^*\| \qquad \textit{for all } \; x \in \Re^n,$$

where x^* is an arbitrary solution of GCP(F,G).

PROOF. By Lemma 7.2 and the fact that $x^* \in \Re^n$ is a solution of GCP(F,G), we obtain

$$\begin{aligned}
|\min\{F_i(x), G_i(x)\}| &= |\min\{F_i(x), G_i(x)\} - \min\{F_i(x^*), G_i(x^*)\}| \\
&\le \max\{|F_i(x) - F_i(x^*)|, |G_i(x) - G_i(x^*)|\} \\
&\le L\|x - x^*\|
\end{aligned}$$

for all $i \in I$ and all $x \in \Re^n$. □

7.4 LEMMA. *For each $i \in I$ and any solution $x^* \in \Re^n$ of GCP(F,G), we have*

$$[F_i(x^*) - F_i(x) + r_i(x)]\,[G_i(x^*) - G_i(x) + r_i(x)] \le 0 \qquad \textit{for all } x \in \Re^n.$$

PROOF. First observe that

$$\begin{aligned}
& [F_i(x^*) - F_i(x) + r_i(x)]\,[G_i(x^*) - G_i(x) + r_i(x)] \\
= \; & [F_i(x^*) - (F_i(x) - G_i(x))_+]\,[G_i(x^*) - (G_i(x) - F_i(x))_+] \\
= \; & F_i(x^*)G_i(x^*) - F_i(x^*)(G_i(x) - F_i(x))_+ - G_i(x^*)(F_i(x) - G_i(x))_+ \\
& + (F_i(x) - G_i(x))_+(G_i(x) - F_i(x))_+,
\end{aligned}$$

where $(\,\cdot\,)_+ = \max\{0, \cdot\}$. Since x^* is a solution of GCP(F,G), we have

$$\begin{aligned}
F_i(x^*)G_i(x^*) &= 0, \\
F_i(x^*)(G_i(x) - F_i(x))_+ &\ge 0, \\
G_i(x^*)(F_i(x) - G_i(x))_+ &\ge 0.
\end{aligned}$$

Also it is easy to see that

$$(F_i(x) - G_i(x))_+(G_i(x) - F_i(x))_+ = 0.$$

Combining the above equalities and inequalities, we get the desired inequality. □

7.5 LEMMA. *Assume that $F, G : \Re^n \to \Re^n$ are Lipschitz–continuous with (common) constant $L > 0$. Then, for each $i \in I$ and any solution $x^* \in \Re^n$ of GCP(F,G), we have*

$$(F_i(x) - F_i(x^*))(G_i(x) - G_i(x^*)) \le 2L\,|r_i(x)|\,\|x - x^*\| \qquad \textit{for all } x \in \Re^n.$$

PROOF. By Lemma 7.4, we have

$$\begin{aligned} 0 &\geq [F_i(x^*) - F_i(x) + r_i(x)]\,[G_i(x^*) - G_i(x) + r_i(x)] \\ &= (F_i(x^*) - F_i(x))(G_i(x^*) - G_i(x)) \\ &\quad + r_i(x)(F_i(x^*) - F_i(x) + G_i(x^*) - G_i(x)) + r_i(x)^2. \end{aligned}$$

This implies

$$\begin{aligned} (F_i(x^*) - F_i(x))(G_i(x^*) - G_i(x)) &\leq r_i(x)\,(F_i(x) - F_i(x^*) + G_i(x) - G_i(x^*)) \\ &\leq |r_i(x)|\,(|F_i(x) - F_i(x^*)| + |G_i(x) - G_i(x^*)|) \\ &\leq 2L\,|r_i(x)|\,\|x - x^*\|, \end{aligned}$$

where the last inequality follows from the assumed Lipschitz–continuity of F and G. □

7.6 LEMMA. *For each $i \in I$, we have*

$$\frac{1}{3+2\sqrt{2}} r_i(x)^2 \leq \varphi(F_i(x), G_i(x)) \leq (3+2\sqrt{2}) r_i(x)^2 \qquad \text{for all } x \in \Re^n.$$

PROOF. Immediate from the definitions of r_i and φ, and Lemma 3.1 in Tseng [30]. □

The following theorems demonstrate that the merit function Ψ actually provides global error bounds for GCP(F,G) under appropriate conditions. Recall that Lemma 5.1 ensures that, under condition (10), the solution of GCP(F,G) is unique whenever the solution set is nonempty.

7.7 THEOREM. *Assume that $F, G : \Re^n \to \Re^n$ are Lipschitz–continuous with constant $L > 0$. Suppose that the solution set X^* of GCP(F,G) is nonempty. Then the inequality*

$$\Psi(x) \leq (3+2\sqrt{2}) n L^2\, dist(x, X^*)^2$$

holds for all $x \in \Re^n$.

PROOF. By Lemma 7.3, we have

$$|r_i(x)| \leq L\, dist(x, X^*).$$

Combining this inequality with Lemma 7.6 and the definition of Ψ, we have

$$\begin{aligned} \Psi(x) &\leq (3+2\sqrt{2}) \sum_{i\in I} r_i(x)^2 \\ &\leq (3+2\sqrt{2}) n L^2\, dist(x, X^*)^2. \end{aligned}$$

This completes the proof. □

7.8 THEOREM. *Assume that $F, G : \Re^n \to \Re^n$ are Lipschitz–continuous with constant $L > 0$ and satisfy condition (10). Suppose that the solution set X^* of GCP(F, G) is nonempty. Then the inequality*

$$\Psi(x) \geq \frac{\rho^2}{4(3+2\sqrt{2})L^2}\|x - x^*\|^2$$

holds for all $x \in \Re^n$, where x^* is the (unique) solution of GCP(F, G).

PROOF. By Lemma 3.1 (b), we have

$$\varphi(F_i(x), G_i(x)) \geq 0$$

for all $i \in I$. Moreover, by Lemma 7.5 and condition (10), there exists at least one index i such that

$$|r_i(x)| \geq \frac{\rho}{2L}\|x - x^*\|,$$

which together with Lemma 7.6 implies that

$$\varphi(F_i(x), G_i(x)) \geq \frac{\rho^2}{4(3+2\sqrt{2})L^2}\|x - x^*\|^2.$$

From this, we obtain the desired inequality for Ψ. □

7.9 COROLLARY. *Under the assumptions of Theorem 7.8, the function $\sqrt{\Psi}$ provides (lower and upper) global error bounds for GCP(F, G).*

PROOF. Immediately follows from Lemma 7.3, Lemma 7.6 and Theorem 7.8. □

Observe that, by (16) and Lemma 7.6, there exist positive constants c_1, c_2, c_3, c_4 such that

$$\begin{aligned} c_1\|r(x)\| \leq \sqrt{M(x;\alpha)} \leq c_2\|r(x)\| \quad &\text{for all } x \in \Re^n, \\ c_3\|r(x)\| \leq \sqrt{\Psi(x)} \leq c_4\|r(x)\| \quad &\text{for all } x \in \Re^n, \end{aligned}$$

where M is Mangasarian–Solodov's implicit Lagrangian with parameter $\alpha > 1$. These inequalities imply that $\sqrt{\Psi}, \sqrt{M}$ and $\|r\|$ are equivalent in the sense that if one of them is a (global) error bound for GCP(F, G) under some assumptions, then the other two also provide (global) error bounds for GCP(F, G) under the same assumptions. The following two corollaries immediately follow from this observation.

7.10 COROLLARY. *Under the assumptions of Theorem 7.8, the function $\sqrt{M}$ with $\alpha > 1$ provides (lower and upper) global error bounds for GCP(F, G).* □

7.11 COROLLARY. *Under the assumptions of Theorem 7.8, the natural residual $\|r\|$ provides (lower and upper) global error bounds for GCP(F, G).* □

For NCP(F), Mangasarian and Ren [21] have shown that, if F is Lipschitz–continuous and strongly monotone, then the natural residual $\|r\|$ provides lower and upper global error bounds. Recall that, in the case of NCP(F), condition (10) is equivalent to the condition of a uniform P-function, which is weaker than the strong monotonicity condition. So Corollary 7.11 improves upon the result by Mangasarian and Ren [21]. On the other hand, Harker and Pang [11, Proposition 4.4 (b)] state that the natural residual provides a global error bound for NCP(F), if F is a Lipschitz–continuous uniform P-function. (In [11, Proposition 4.4 (b)], the Lipschitz–continuity assumption is missing but is actually needed [28].) The proof of that proposition is attributed to [26], but there a proof is explicitly given for LCP(q, M) only. The proof for NCP(F), however, goes through along the same lines ([28]).

8 Final Remarks

We have proposed an unconstrained minimization reformulation of the generalized complementarity problem GCP(F, G). Several properties of our merit function were proved which are important from both the theoretical and numerical points of view. However, we have not mentioned any algorithm for minimizing the merit function Ψ. Since Ψ is once (but not twice) continuously differentiable, it seems reasonable to use a first–order method for minimizing Ψ. Moreover, the recent research of Qi and Jiang [29] on constrained optimization problems and those of Facchinei and Soares [2] and Jiang and Qi [13] on nonlinear complementarity problems suggest that it should also be possible to apply a second–order (generalized Newton) method for minimizing Ψ. This could be of interest for future research work.

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