Equivalent Unconstrained Minimization and Global Error Bounds for Variational Inequality Problems[†]

by

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Abstract. New merit functions for variational inequality problems are constructed through the Moreau-Yosida regularization of some gap functions. The proposed merit functions constitute *unconstrained* optimization problems equivalent to the original variational inequality problem under suitable assumptions. Conditions are studied for those merit functions to be differentiable and for any stationary point of those functions to be a solution of the original variational inequality problem. Moreover, those functions are shown to have such a remarkable property that they provide global error bounds for general variational inequality problems under the strong monotonicity assumption only.

Key words. Variational inequality problem, merit function, unconstrained optimization problem, global error bounds.

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1. Introduction

Merit functions for variational inequality and complementarity problems have recently drawn much attention. Such a function naturally constitutes an equivalent optimization formulation and has turned out to be very useful in designing new globally convergent algorithms and analyzing the rate of convergence of some algorithms [2, 5–8, 10–15, 17–20, 22–29]. For nonlinear complementarity problems, both constrained and unconstrained differentiable optimization formulations are already known and various interesting results on iterative methods and error bounds have been established [7, 10, 15, 17, 26, 27, 29]. For variational inequality problems, constrained differentiable optimization formulations have been proposed [5, 6, 11, 18, 24, 28]. To the authors' knowledge, however, an unconstrained differentiable optimization formulation has yet to be discussed in the literature. The purpose of this paper is to present unconstrained differentiable optimization formulations for general variational inequality problems and give some error bound results based on them.

The variational inequality problem (VIP) is to find a vector x in a closed convex subset S of R^n such that

$$\langle F(x), y - x\rangle \geq 0 \quad \text{for all } y \in S,$$

where F is a mapping from R^n into itself and $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n. Throughout the paper, we shall assume that the mapping F is continuous.

First, we review some well-known merit functions for VIP. Let $\Psi : R^n \times R^n \to R$ be defined by

$$\Psi(x, y) = \langle F(x), x - y\rangle. \tag{1}$$

Using Ψ, define the class of functions $f(\,\cdot\,;\alpha) : R^n \to R \cup \{+\infty\}$ by

$$f(x;\alpha) = \sup_{y \in S} \left\{ \Psi(x, y) - \alpha \|x - y\|^2 \right\}, \tag{2}$$

where α is a nonnegative parameter. This function was first introduced by Auslender [2] for $\alpha = 0$ (see also [9]) and by Fukushima [6] for $\alpha > 0$ (see also [1, 11]). The function $f(\,\cdot\,;0)$ is commonly called the gap function, while the function $f(\,\cdot\,;\alpha)$ with $\alpha > 0$ is called the regularized gap function. For each $\alpha \geq 0$, the function $f(\,\cdot\,;\alpha)$ is nonnegative on S and becomes

zero at any solution of VIP [6, 9]. Hence, VIP is equivalent to finding a global minimizer of $f(\,\cdot\,;\alpha)$ on S. If F is a continuous mapping, then $f(\,\cdot\,;\alpha)$ is lower semi-continuous for $\alpha = 0$ [4, Lemma 4.1] and continuous for $\alpha > 0$ [6, Theorem 3.1]. The gap function $f(\,\cdot\,;0)$ has the serious drawback, however, that it is in general nondifferentiable even if F is differentiable, and even worse, it may not be finite-valued. On the other hand, the regularized gap function $f(\,\cdot\,;\alpha)$ with $\alpha > 0$ has such desirable properties that it is finite-valued everywhere and is differentiable whenever F is differentiable [6, Theorem 3.1]. (See [6, 11] for further properties of the regularized gap function.) A modification of the regularized gap function has recently been proposed in [22, 23].

Closely related to the functions $f(\,\cdot\,;\alpha)$ is the class of functions $h(\,\cdot\,;\beta) : R^n \to R \cup \{+\infty\}$ given by

$$h(x;\beta) = \sup_{y \in S} \left\{ -\Psi(y,x) + \beta \|x-y\|^2 \right\}, \tag{3}$$

where β is a nonnegative parameter. This function has been studied in [18] for the case $\beta = 0$. In particular, if F is pseudo-monotone on S, i.e.,

$$\langle F(y), x-y \rangle \geq 0 \Longrightarrow \langle F(x), x-y \rangle \geq 0 \quad \text{for all } x, y \in S,$$

then the function $h(\,\cdot\,;0)$ is nonnegative on S and vanishes at any solution of VIP, implying that VIP is equivalent to minimizing $h(\,\cdot\,;0)$ over S [18, Proposition 1]. We will see that this property is extended to the case $\beta > 0$. It is easily seen that, for any $\beta \geq 0$, the function $h(\,\cdot\,;\beta)$ is convex. Moreover, if F is continuous, then $h(\,\cdot\,;\beta)$ is lower semi-continuous, as shown in the next section.

It should be noted that the above-mentioned merit functions all constitute constrained optimization problems equivalent to VIP. In this paper, we explore the possibility of constructing unconstrained differentiable optimization problems equivalent to VIP. Specifically, we consider the following functions derived from the Moreau-Yosida regularization:

$$\phi_f(x;\alpha,\lambda) \;=\; \inf_{z \in S} \left\{ f(z;\alpha) + \lambda \|x-z\|^2 \right\} \tag{4}$$

and

$$\phi_h(x;\beta,\lambda) \;=\; \inf_{z \in S} \left\{ h(z;\beta) + \lambda \|x-z\|^2 \right\}, \tag{5}$$

where λ is a positive parameter, and $f(\,\cdot\,;\alpha)$ and $h(\,\cdot\,;\beta)$ are defined by (2) and (3), respectively. In practice, the functions $\phi_f(\,\cdot\,;\alpha,\lambda)$ and $\phi_h(\,\cdot\,;\beta,\lambda)$ may not be easy to evaluate. Nevertheless they enjoy some nice theoretical properties that existing merit functions for VIP do not have. Under some assumptions, the functions $\phi_f(\,\cdot\,;\alpha,\lambda)$ and $\phi_h(\,\cdot\,;\beta,\lambda)$ are shown to be differentiable even if F is not differentiable. We shall investigate conditions under which any stationary point of these function solves the original VIP. We also discuss how the foregoing results are specialized to affine VIP. Finally, we will show that $\phi_f(\,\cdot\,;\alpha,\lambda)$ and $\phi_h(\,\cdot\,;\beta,\lambda)$ provide global error bounds for the strongly monotone VIP. We remark that those results are valid for problems with general convex constraints S and do not rely on the Lipschitz continuity of F, while the well-known error bound for VIP obtained by Pang [19] assumes that the set S is polyhedral and F is both strongly monotone and Lipschitz continuous.

2. VIP as unconstrained minimization

In this section, we show that the unconstrained minimization of the function $\phi_f(\,\cdot\,;\alpha,\lambda)$ defined by (4) is equivalent to VIP, and that, under some additional assumptions, the unconstrained minimization of the function $\phi_h(\,\cdot\,;\beta,\lambda)$ defined by (5) is also equivalent to VIP. Further, we shall investigate the differentiability and the convexity of these functions. Recall that the mapping $F : R^n \to R^n$ is said to be monotone on S if

$$\langle F(x) - F(y), x - y\rangle \geq 0 \quad \text{for all } x, y \in S,$$

and strongly monotone with modulus $\mu > 0$ on S if

$$\langle F(x) - F(y), x - y\rangle \geq \mu\|x - y\|^2 \quad \text{for all } x, y \in S.$$

First, we state some lemmas, which will be useful in proving the subsequent theorems.

Lemma 2.1 *For any $\alpha \geq 0$, $f(\,\cdot\,;\alpha)$ is nonnegative on S. Moreover, $f(x;\alpha) = 0$ if and only if x is a solution of VIP.*

Proof. See [9] for the case $\alpha = 0$ and [6, Theorem 3.1] for the case $\alpha > 0$. □

Lemma 2.2 *For any $\beta \geq 0$, the function $h(\,\cdot\,;\beta)$ is a closed convex function.*

Proof. The convexity of $h(\,\cdot\,;\beta)$ follows from the definition (3) directly, since $-\Psi(y,\,\cdot\,)+\beta\|\,\cdot\,-y\|^2$ is convex for every y. The closedness of $h(\,\cdot\,;\beta)$ follows from the fact that a pointwise supremum of continuos functions yields a lower semicontinuous function. □

Lemma 2.3 *(a) For any $\beta \geq 0$, $h(\,\cdot\,;\beta)$ is nonnegative on S.*

(b) Suppose that F is pseudo-monotone on S. Then x is a solution of VIP if and only if $h(x;0) = 0$ and $x \in S$.

(c) Suppose that F is strongly monotone on S with modulus μ and that β is chosen sufficiently small to satisfy $0 \leq \beta \leq \mu$. Then x is a solution of VIP if and only if $h(x;\beta) = 0$ and $x \in S$.

Proof. (a) Obvious from the definition (3) of $h(\,\cdot\,;\beta)$. (b) See [18, Proposition 1]. (c) By the definition (1) of Ψ, we have

$$\begin{aligned} \Psi(x,y) &\geq -\Psi(y,x) + \mu\|x-y\|^2 \\ &\geq -\Psi(y,x) + \beta\|x-y\|^2 \\ &\geq -\Psi(y,x) \qquad \text{for all } x, y \in S, \end{aligned}$$

implying that

$$f(x;0) \geq h(x;\beta) \geq h(x;0) \quad \text{for all } x \in S. \tag{6}$$

Let x^* be a solution of VIP. Then, since $f(x^*;0) = 0$ by Lemma 2.1 and $h(x^*;0) = 0$ by part (b) of this lemma, we have $h(x^*;\beta) = 0$ by (6). Conversely, let $h(x;\beta) = 0$. Then, since $h(\,\cdot\,;0)$ is nonnegative on S by part (a), it follows from (6) that $h(x;0) = 0$. Thus, x is a solution of VIP by part (b). □

Using these lemmas, we establish the equivalence between VIP and the unconstrained minimization of $\phi_f(\,\cdot\,;\alpha,\lambda)$ and $\phi_h(\,\cdot\,;\beta,\lambda)$. The next theorem shows that these functions are

nonnegative on R^n and that, under suitable conditions, the sets of unconstrained minima of these functions coincide with the set of solutions of VIP.

Theorem 2.1 *(a) For any $\alpha \geq 0$, $\beta \geq 0$ and $\lambda > 0$, the functions $\phi_f(\,\cdot\,;\alpha,\lambda)$ and $\phi_h(\,\cdot\,;\beta,\lambda)$ are nonnegative on R^n.*

(b) For any $\alpha \geq 0$ and $\lambda > 0$, x^* is a solution of VIP if and only if $\phi_f(x^*;\alpha,\lambda) = 0$, i.e., $\phi_f(\,\cdot\,;\alpha,\lambda)$ attains its global minimum at x^*.

(c) Suppose that F is pseudo-monotone on S. Then, for any $\lambda > 0$, x^* is a solution of VIP if and only if $\phi_h(x^*;0,\lambda) = 0$, i.e., $\phi_h(\,\cdot\,;0,\lambda)$ attains its global minimum at x^*.

(d) Suppose that F is strongly monotone on S with modulus μ and that β is sufficiently small to satisfy $0 \leq \beta \leq \mu$. Then, for any $\lambda > 0$, x^* is a solution of VIP if and only if $\phi_h(x^*;\beta,\lambda) = 0$, i.e., $\phi_h(\,\cdot\,;\beta,\lambda)$ attains its global minimum at x^*.

Proof. We consider the function $\phi_f(\,\cdot\,;\alpha,\lambda)$. (a) Since $f(\,\cdot\,;\alpha)$ is nonnegative on S by Lemma 2.1, we can easily deduce from the definition (4) of $\phi_f(\,\cdot\,;\alpha,\lambda)$ that $\phi_f(x;\alpha,\lambda)$ is nonnegative for all $x \in R^n$. (b) Suppose that x^* is a solution of VIP. Then, we have

$$\begin{aligned} \phi_f(x^*;\alpha,\lambda) &= \inf_{z\in S}\left\{f(z;\alpha) + \lambda\|x^* - z\|^2\right\} \\ &\leq f(x^*;\alpha) + \lambda\|x^* - x^*\|^2 \\ &= 0, \end{aligned}$$

where the last equality follows from $f(x^*;\alpha) = 0$. Since $\phi_f(x;\alpha,\lambda) \geq 0$ for all x as shown above, we obtain $\phi_f(x^*;\alpha,\lambda) = 0$. Conversely, suppose $\phi_f(x^*;\alpha,\lambda) = 0$. Then since $f(z;\alpha) \geq 0$ for all $z \in S$, it follows from the definition (4) of $\phi_f(\,\cdot\,;\alpha,\lambda)$ that there exists a sequence $\{z^k\}$ in S such that $f(z^k;\alpha) \to 0$ and $\|z^k - x^*\| \to 0$. Since S is closed, $z^k \to x^*$ and $z^k \in S$ imply $x^* \in S$. Moreover, since $f(\,\cdot\,;\alpha)$ is lower semi-continuous for $\alpha = 0$ [4, 11] and continuous for $\alpha > 0$ [6], we have

$$0 \leq f(x^*;\alpha) \leq \lim_{k\to\infty} f(z^k;\alpha) = 0.$$

Thus, x^* must be a solution of VIP.

By using Lemmas 2.2 and 2.3, the proof for the function $\phi_h(\,\cdot\,;\beta,\lambda)$ can be done analogously. □

By this theorem, the unconstrained minimization problem

$$\min_{x\in R^n} \quad \phi_f(x;\alpha,\lambda) \tag{7}$$

is equivalent to VIP, and the unconstrained minimization problem

$$\min_{x\in R^n} \quad \phi_h(x;\beta,\lambda) \tag{8}$$

is equivalent to VIP under suitable assumptions on F and the parameters involved.

In order for these minimization problems to be practically useful, it is desirable that the objective functions $\phi_f(\,\cdot\,;\alpha,\lambda)$ and $\phi_h(\,\cdot\,;\beta,\lambda)$ are everywhere differentiable. For the discussions to follow, it will be convenient to define the functions $\Phi_f(\,\cdot\,,\,\cdot\,;\alpha,\lambda) : R^{2n} \to R\cup\{+\infty\}$ and $\Phi_h(\,\cdot\,,\,\cdot\,;\beta,\lambda) : R^{2n} \to R\cup\{+\infty\}$ by

$$\Phi_f(x,z;\alpha,\lambda) = f(z;\alpha) + \lambda\|x-z\|^2$$

and

$$\Phi_h(x,z;\beta,\lambda) = h(z;\beta) + \lambda\|x-z\|^2,$$

respectively. By (4) and (5), we have

$$\phi_f(x;\alpha,\lambda) = \inf_{z\in S} \Phi_f(x,z;\alpha,\lambda)$$

and

$$\phi_h(x;\beta,\lambda) = \inf_{z\in S} \Phi_h(x,z;\beta,\lambda).$$

Whenever the functions $\Phi_f(x,\,\cdot\,;\alpha,\lambda)$ and $\Phi_h(x,\,\cdot\,;\beta,\lambda)$ are assumed to attain their minima uniquely in the set S, we shall denote those minima by $z_f(x;\alpha,\lambda)$ and $z_h(x;\beta,\lambda)$, respectively. The following propositions are consequences of [2, Chap. 4, Theorem 1.7]. Note that these propositions do not rely upon the differentability of F.

Proposition 2.1 *Let $\alpha \geq 0$ and $\lambda > 0$. If the function $\Phi_f(x,\,\cdot\,;\alpha,\lambda)$ attains its unique minimum $z_f(x;\alpha,\lambda)$ on S for each $x \in R^n$, then $\phi_f(\,\cdot\,;\alpha,\lambda)$ is differentiable on R^n and*

$$\nabla\phi_f(x;\alpha,\lambda) = 2\lambda\left(x - z_f(x;\alpha,\lambda)\right).$$

Proof. Immediate from Theorem 1.7 in [2, Chap. 4]. □

Proposition 2.2 *For any $\beta \geq 0$ and $\lambda > 0$, the function $\Phi_h(x, \cdot\,; \beta, \lambda)$ attains its minimum $z_h(x; \beta, \lambda)$ uniquely. Moreover, $\phi_h(\cdot\,; \beta, \lambda)$ is a differentiable convex function on R^n and*

$$\nabla \phi_h(x; \beta, \lambda) = 2\lambda \left(x - z_h(x; \beta, \lambda)\right).$$

Proof. By Lemma 2.2, $h(\cdot\,; \beta)$ is a closed convex function. Thus, by Theorem 31.5 in [21], $\Phi_h(x, \cdot\,; \beta, \lambda)$ uniquely attains its minimum on S for any x and hence, by Theorem 1.7 in [2, Chap. 4], $\phi_h(\cdot\,; \beta, \lambda)$ is differentiable and its gradient is represented as indicated in the proposition. Moreover, it follows from the convexity of $h(\cdot\,; \beta)$ that $\phi_h(\cdot\,; \beta, \lambda)$ is also convex (see the proof of Proposition 4.1 in [3]). □

Theorem 2.1 is concerned with the equivalence between VIP and global minimization of $\phi_f(\cdot\,; \alpha, \lambda)$ and $\phi_h(\cdot\,; \beta, \lambda)$. In general, however, there may exist local minima or stationary points which do not solve VIP. In the remainder of the section, we study conditions under which any stationary point of problems (7) and (8) is a solution of VIP.

The following proposition gives a condition for $\phi_f(\cdot\,; 0, \lambda)$ to be convex.

Proposition 2.3 *Suppose that $\Psi(\cdot\,, y)$ is convex for each fixed $y \in S$. Then, for any $\lambda > 0$, $\phi_f(\cdot\,; 0, \lambda)$ is a differentiable convex function on R^n.*

Proof. Since $\Psi(\cdot\,, y)$ is finite-valued and convex for each fixed $y \in S$, $f(\cdot\,; 0)$ is a closed proper convex function. So we can prove the convexity of $\phi_f(\cdot\,; 0, \lambda)$ in a way similar to the proof of Proposition 4.1 in [3]. Moreover, since $\Phi_f(x, \cdot\,; 0, \lambda)$ uniquely attains its minimum on S for any x by Theorem 31.5 in [21], $\phi_f(\cdot\,; 0, \lambda)$ is differentiable by Proposition 2.1. □

Note that it is shown in [9] that if S is polyhedral convex, $\langle F(x), x \rangle$ is convex on S and each component of F is concave on S, then $f(\cdot\,; 0)$ is convex. Thus it follows that, under these assumptions, $\phi_f(\cdot\,; 0, \lambda)$ is a differentiable convex function.

The following result is a consequence of Proposition 2.3.

Theorem 2.2 *Suppose that $\Psi(\,\cdot\,, y)$ is convex for each fixed $y \in S$. Then, for each $\lambda > 0$, any stationary point of $\phi_f(\,\cdot\,; 0, \lambda)$ is a solution of VIP.*

Proof. Since $\phi_f(\,\cdot\,; 0, \lambda)$ is a differentiable convex function by Proposition 2.1, $\nabla \phi_f(x; 0, \lambda) = 0$ if and only if $\phi_f(\,\cdot\,; 0, \lambda)$ attains its global minimum at x. So the result follows from Theorem 2.1. □

Next we consider the function $\phi_f(\,\cdot\,; \alpha, \lambda)$ with $\alpha > 0$. The following theorem gives conditions under which any stationary point of $\phi_f(\,\cdot\,; \alpha, \lambda)$ is a solution of VIP.

Theorem 2.3 *Let $\alpha > 0$ and $\lambda > 0$. Suppose that the function $\Phi_f(x, \,\cdot\,; \alpha, \lambda)$ attains its unique minimum $z_f(x; \alpha, \lambda)$ on S for any fixed $x \in R^n$. If F is differentiable and $\nabla F(x)$ is positive definite on S, then any stationary point of $\phi_f(\,\cdot\,; \alpha, \lambda)$ is a solution of VIP.*

Proof. Let $\hat{x}$ be an arbitrary stationary point of $\phi_f(\,\cdot\,; \alpha, \lambda)$. Then, by Proposition 2.1, it satisfies

$$\nabla \phi_f(\hat{x}; \alpha, \lambda) = 2\lambda \left(\hat{x} - z_f(\hat{x}; \alpha, \lambda)\right) = 0.$$

Hence, we have $\hat{x} = z_f(\hat{x}; \alpha, \lambda) \in S$. On the other hand, since $z_f(x; \alpha, \lambda)$ is the minimizer of $\Phi_f(x, \,\cdot\,; \alpha, \lambda)$ on S and since $f(\,\cdot\,; \alpha)$ is differentiable [6], the first-order optimality condition yields

$$\langle \nabla_z \Phi_f(x, z_f(x; \alpha, \lambda); \alpha, \lambda), y - z_f(x; \alpha, \lambda) \rangle \geq 0 \quad \text{for all } y \in S, \tag{9}$$

that is,

$$\langle \nabla f(z_f(x; \alpha, \lambda); \alpha) + 2\lambda(z_f(x; \alpha, \lambda) - x), y - z_f(x; \alpha, \lambda) \rangle \geq 0 \quad \text{for all } y \in S. \tag{10}$$

Substituting $\hat{x}$ for x in (10) and using $\hat{x} = z_f(\hat{x}; \alpha, \lambda)$, we have

$$\langle \nabla f(\hat{x}; \alpha), y - \hat{x} \rangle \geq 0 \quad \text{for all } y \in S.$$

Namely, $\hat{x}$ is a stationary point of $f(\,\cdot\,; \alpha)$ on S. Then, by Theorem 3.3 in [6], the positive definiteness of ∇F ensures that $\hat{x}$ is a solution of VIP. □

Finally, we consider the function $\phi_h(\,\cdot\,; \beta, \lambda)$. Proposition 2.2 and Theorem 2.1 immediately yield the following theorem.

Theorem 2.4 *Let $\lambda > 0$. If F is pseudo-monotone on S, then any stationary point of $\phi_h(\,\cdot\,; 0, \lambda)$ is a solution of VIP. Moreover, if F is strongly monotone on S with modulus μ and if β is chosen sufficiently small to satisfy $0 \le \beta \le \mu$, then any stationary point of $\phi_h(\,\cdot\,; \beta, \lambda)$ is a solution of VIP.*

Proof. First note that, for each $\beta \ge 0$ and $\lambda > 0$, $\phi_h(\,\cdot\,; \beta, \lambda)$ is a differentiable convex function by Proposition 2.2. Thus, $\nabla\phi_h(x; \beta, \lambda) = 0$ if and only if $\phi_h(\,\cdot\,; \beta, \lambda)$ attains its global minimum at x. The theorem then follows from Theorem 2.1 (c) and (d). □

3. Monotone affine VIP

In this section, we consider the special case of VIP, where F is the affine mapping

$$F(x) = Mx + q.$$

Here the $n \times n$ matrix M is assumed to be positive semidefinite, that is, the mapping F is monotone. The next two theorems show some properties of $\phi_f(\,\cdot\,; \alpha, \lambda)$ and $\phi_h(\,\cdot\,; \beta, \lambda)$ for the monotone affine VIP.

Theorem 3.1 *Let μ be a nonnegative constant such that*

$$\langle x, Mx \rangle \ge \mu \|x\|^2 \quad \text{for all } x \in R^n.$$

Then, for any $\lambda > 0$ and $\alpha \ge 0$ such that $\alpha \le \mu$, $\phi_f(\,\cdot\,; \alpha, \lambda)$ is a differentiable convex function. Moreover, any stationary point of $\phi_f(\,\cdot\,; \alpha, \lambda)$ is a solution of the affine VIP.

Proof. If $0 \le \alpha \le \mu$, then the function

$$\Psi(z, y) - \alpha\|z - y\|^2 = \langle Mz + q, z - y \rangle - \alpha\|z - y\|^2$$

is convex in z for each fixed y. Hence, by the definition (2), $f(\,\cdot\,; \alpha)$ is a closed convex function. So it follows from the proof of Proposition 4.1 in [3] that $\phi_f(\,\cdot\,; \alpha, \lambda)$ is a closed convex function. Moreover, by Theorem 31.5 in [21], $\Phi_f(x, \,\cdot\,; \alpha, \lambda)$ uniquely attains its minimum on S for any x. Hence, by Proposition 2.1, $\phi_f(\,\cdot\,; \alpha, \lambda)$ is differentiable. This proves the first half of the theorem. Since $\nabla\phi_f(x; \alpha, \lambda) = 0$ if and only if x is a global minimizer of $\phi_f(\,\cdot\,; \alpha, \lambda)$, the last half of the theorem follows from Theorem 2.1 (b). □

Theorem 3.2 *Let μ be a nonnegative constant such that*

$$\langle x, Mx \rangle \geq \mu \|x\|^2 \quad \text{for all } x \in R^n.$$

Then, for any $\lambda > 0$ and $\beta \geq 0$ such that $\beta \leq \mu$, $\phi_h(\,\cdot\,;\beta,\lambda)$ is a differentiable convex function and any stationary point of $\phi_h(\,\cdot\,;\beta,\lambda)$ is a solution of the affine VIP.

Proof. By Proposition 2.2 and Theorem 2.4, we get the desired results using a similar argument to the proof of Theorem 3.1. □

Note that if M is not positive definite but simply positive semidefinite, then the constant μ in Theorems 3.1 and 3.2 is zero, and hence α and β must be zero, too. Namely, the theorems then apply to the functions $\phi_f(\,\cdot\,;0,\lambda)$ and $\phi_h(\,\cdot\,;0,\lambda)$.

4. Global error bounds

Error bounds play an important role in analyzing the convergence rate of iterative algorithms [14, 25]. In particular, the natural residual $r : R^n \to R$, which is defined by

$$r(x) = \|x - [x - F(x)]_S^+\|, \tag{11}$$

where $[\,\cdot\,]_S^+$ denotes the orthogonal projection onto S, is known to provide a global error bound for linearly constrained VIP under the Lipschitz continuity and strong monotonicity of F [19]. In this section, we show that $\phi_f(\,\cdot\,;\alpha,\lambda)$ and $\phi_h(\,\cdot\,;\beta,\lambda)$ provide global error bounds for VIP under the strong monotonicity of F.

To begin with, we point out that, when F is strongly monotone, the functions $f(\,\cdot\,;\alpha)$ and $h(\,\cdot\,;\beta)$ provide error bounds on S, provided that the parameters α and β are small enough. Note that the strong monotonicity of F guarantees the existence of a unique solution to VIP.

Lemma 4.1 *If F is strongly monotone on S with modulus μ and if α is chosen sufficiently small to satisfy $0 \leq \alpha < \mu$, then we have $f(x^*;\alpha) = 0$ and*

$$f(x;\alpha) \quad \geq \quad (\mu - \alpha)\,\|x - x^*\|^2 \qquad \text{for all } x \in S,$$

where x^* is the unique solution of VIP.

Proof. See Proposition 3.4 in [24] and Theorem 4.5 in [11]. □

Lemma 4.2 *If F is strongly monotone on S with modulus μ and if β is chosen sufficiently small to satisfy $0 < \beta \leq \mu$, then we have $h(x^*; \beta) = 0$ and*

$$h(x; \beta) \quad \geq \quad \beta \|x - x^*\|^2 \qquad \text{for all } x \in S,$$

where x^* is the unique solution of VIP.

Proof. By Lemma 2.3, we have $h(x^*, \beta) = 0$. Let $x \in S$ be arbitrary. Then, we have

$$\begin{aligned} h(x; \beta) &= \sup_{y \in S} \left\{ \langle F(y), x - y \rangle + \beta \|x - y\|^2 \right\} \\ &\geq \langle F(x^*), x - x^* \rangle + \beta \|x - x^*\|^2 \\ &\geq \beta \|x - x^*\|^2, \end{aligned}$$

where the last inequality follows from the fact that x^* is a solution of VIP. □

Note that the error bounds given in the above lemmas are valid on the constraint set S. Using these lemmas, we prove below that $\phi_f(\,\cdot\,; \alpha, \lambda)$ and $\phi_h(\,\cdot\,; \beta, \lambda)$ provide global error bounds for VIP on the whole space R^n, whenever F is strongly monotone.

The following geometric lemma will be used to prove the main theorems in this section.

Lemma 4.3 *Let x be an arbitrary point in R^n and let y be an arbitrary point in S. Then we have*

$$\|[z]_S^+ - x\|^2 \geq \|z - x\|^2,$$

where $z = (x + y)/2$.

Proof. We have

$$\begin{aligned} \|[z]_S^+ - x\|^2 - \|z - x\|^2 &= \langle [z]_S^+ - z, [z]_S^+ + z - 2x \rangle \\ &= \|[z]_S^+ - z\|^2 + 2\langle [z]_S^+ - z, z - x \rangle \\ &\geq 2\langle [z]_S^+ - z, z - x \rangle. \end{aligned}$$

Since $x = 2z - y$, we get

$$\begin{aligned} \langle [z]_S^+ - z, z - x \rangle &= \langle [z]_S^+ - z, y - z \rangle \\ &\geq 0, \end{aligned}$$

where the inequality follows from the Projection Theorem in [3, page 221]. □

The next theorem shows that the growth rate of $\phi_f(\,\cdot\,;\alpha,\lambda)$ is in the order of the squared distance to the unique solution of VIP, provided that the parameter α is chosen sufficiently small.

Theorem 4.1 *If F is strongly monotone on S with modulus μ and if α is chosen sufficiently small to satisfy $0 \leq \alpha < \mu$, then, for each $\lambda > 0$,*

$$\frac{1}{4}\min\{\mu - \alpha, \lambda\}\,\|x - x^*\|^2 \leq \phi_f(x;\alpha,\lambda) \leq \lambda\|x - x^*\|^2 \quad \text{for all } x \in R^n,$$

where x^* is the unique solution of VIP.

Proof. First we consider the right-hand inequality. Since $x^* \in S$ and $f(x^*;\alpha) = 0$ by Lemma 2.1, we have

$$\begin{aligned} \phi_f(x;\alpha,\lambda) &= \inf_{z\in S}\left\{f(z;\alpha) + \lambda\|z - x\|^2\right\} \\ &\leq f(x^*;\alpha) + \lambda\|x - x^*\|^2 \\ &= \lambda\|x - x^*\|^2. \end{aligned}$$

Next, we prove the left-hand inequality. It follows from Lemma 4.1 that

$$\begin{aligned} \phi_f(x;\alpha,\lambda) &= \inf_{z\in S}\left\{f(z;\alpha) + \lambda\|z - x\|^2\right\} \\ &\geq \inf_{z\in S}\left\{(\mu - \alpha)\,\|z - x^*\|^2 + \lambda\|z - x\|^2\right\} \\ &\geq \min\{\mu - \alpha, \lambda\}\inf_{z\in S}\left\{\|z - x^*\|^2 + \|z - x\|^2\right\} \\ &= \min\{\mu - \alpha, \lambda\}\left(\left\|\left[\frac{x + x^*}{2}\right]_S^+ - x^*\right\|^2 + \left\|\left[\frac{x + x^*}{2}\right]_S^+ - x\right\|^2\right) \\ &\geq \min\{\mu - \alpha, \lambda\}\left\|\left[\frac{x + x^*}{2}\right]_S^+ - x\right\|^2 \end{aligned}$$

$$\begin{aligned} &\geq \min\{\mu-\alpha,\lambda\}\left\|\frac{x+x^*}{2}-x\right\|^2 \\ &= \frac{1}{4}\min\{\mu-\alpha,\lambda\}\,\|x-x^*\|^2, \end{aligned}$$

where the last inequality follows from Lemma 4.3. □

Now we turn our attention to the function $\phi_h(\,\cdot\,;\beta,\lambda)$. The next theorem shows that the quadratic growth rate of $\phi_h(\,\cdot\,;\beta,\lambda)$ is ensured, provided that the parameter β is chosen sufficiently small.

Theorem 4.2 *If F is strongly monotone on S with modulus μ and if β is chosen sufficiently small to satisfy $0<\beta\leq\mu$, then, for each $\lambda>0$,*

$$\frac{1}{4}\min\{\beta,\lambda\}\,\|x-x^*\|^2 \leq \phi_h(x;\beta,\lambda) \leq \lambda\|x-x^*\|^2 \quad \textit{for all } x\in R^n,$$

where x^* is the unique solution of VIP.

Proof. Noting that $x^*\in S$ and $h(x^*;\beta)=0$, the right-hand inequality can be proved in a way similar to the first part of the proof of Theorem 4.1. Moreover, by using Lemma 4.2, we can prove the left-hand inequality analogously to the last part of the proof of Theorem 4.1. □

Theorems 4.1 and 4.2 demonstrate that $\phi_f(\,\cdot\,;\alpha,\lambda)$ and $\phi_h(\,\cdot\,;\beta,\lambda)$ provide global error bounds for VIP without the Lipschitz continuity of F. Recall that, for the natural residual r defined by (11) to provide a global error bound, not only the strong monotonicity but also the Lipschitz continuity of F are required [19]. In this respect, $\phi_f(\,\cdot\,;\alpha,\lambda)$ and $\phi_h(\,\cdot\,;\beta,\lambda)$ are more favorable than the natural residual r.

5. Concluding remarks

When S is given by

$$S=\{x\in R^n_+ \mid g_i(x)\leq 0,\ \ i=1,\ldots,m\}, \tag{12}$$

where R^n_+ denotes the nonnegative orthant in R^n and $g_i: R^n\to R$ are twice differentiable convex functions, the Karush-Kuhn-Tucker conditions for VIP yield the nonlinear complementarity

problem: Find $(x, \nu) \in R^n_+ \times R^m_+$ such that

$$\langle (x, \nu), H(x, \nu) \rangle = 0, \quad H(x, \nu) \geq 0, \tag{13}$$

where

$$H(x, \nu) = \begin{bmatrix} F(x) - \nabla g(x)\nu \\ -g(x) \end{bmatrix}$$

with $g(x) = (g_1(x), g_2(x), \ldots, g_m(x))^T$. For nonlinear complementarity problems, various equivalent formulations as a system of equations or an unconstrained optimization problem have been proposed and studied extensively [7, 10, 16, 17, 25, 27, 29]. So we might first reformulate VIP with the inequality constraints (12) as the nonlinear complementarity problem (13), and then convert it into one of the known equivalent unconstrained optimization problems. However, such a transformation into the nonlinear complementality problem (13) is not necessarily a panacea for dealing with the inequality constrained VIP. For instance, global error bounds for nonlinear complementarity problems are obtained under the strong monotonicity of the mapping involved [19, 29]. Unfortunately, it is unlikely that the nonlinear complementarity problem (13) satisfies the strong monotonicity condition, because the Jacobian

$$\nabla H(x, \nu) = \begin{bmatrix} \nabla F(x) + \sum_{i=1}^m \nu_i \nabla^2 g_i(x) & \nabla g(x) \\ -\nabla g(x)^T & 0 \end{bmatrix}$$

is never positive definite for any (x, ν), even if F is strongly monotone. On the other hand, as shown in the preceding sections, the functions $\phi_f(\,\cdot\,; \alpha, \lambda)$ and $\phi_h(\,\cdot\,; \beta, \lambda)$ enjoy various favorable properties, particularly when F is strongly monotone.

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