On Branch Labels of Parallel Components of the L-section Minimal Trellis Diagrams for Binary Linear Block Codes*

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Summary

In an earlier paper[1], we have shown that each section of the L-section minimal trellis diagram for a linear block code consists of parallel and structurally identical (except branch labels) subgraphs without cross connections. These parallel subgraphs are called parallel components of the section. In this paper, it is shown that if the sets of path label sequences of two parallel components have a common sequence, then the parallel components have the same branch labels, and the number of parallel components with the same branch labels in each section and the detail structure of each parallel component up to its branch labels are analyzed and expressed in terms of the dimensions of specific linear codes related to the given code. As an example, the 2^i-section minimal trellis diagram for a Reed-Muller code is analyzed. Complexity measures of soft-decision maximum likelihood decoding for binary linear block codes are also discussed.

Keywords: Linear block code, trellis diagram, parallel structure, and Reed-Muller codes.

1. Introduction

A trellis diagram for a linear block code is generally expressed and constructed in terms of sections[1]−[4]. A binary (N, K) block code C has an N-section trellis diagram [1] in which each branch represents a code bit. Suppose N is a multiple of L. Then an L-section trellis diagram can be constructed from the N-section trellis diagram. In this L-section trellis diagram, each branch represents N/L code bits. In many cases, the trellis diagrams for linear block codes, such as Reed-Muller codes, are loosely connected and have simple parallel structure. The loose connectivity and parallel structure simplify the implementation of Viterbi decoding. In fact, smaller Viterbi decoders can be built to process the parallel sub-trellis diagrams in parallel. This speeds up the decoding process which is important in high-speed data communication systems.

The parallel structure of the L-section minimal trellis diagram for a binary linear block code was first investigated in [1], in which the number of parallel and structurally identical (except branch labels) sub-trellis diagrams without cross connections, called parallel components, in each section was expressed in terms of the dimensions of specific linear codes related to the given code. In the same paper, the connectivity of parallel components between adjacent sections was also completely analyzed.

In this paper, it is shown that if the sets of path label sequences of two parallel components in a section of an L-section minimal trellis diagram for a binary linear block code have a common sequence, then the two parallel components have the same branch labels, and the number of parallel components with the same branch labels and the detail structure of each parallel component up to its branch labels in each section are investigated and expressed in terms of the dimensions of specific linear codes related to the given code. The 8-section and 16-section minimal trellis diagrams for the second- and third-order Reed-Muller codes of length 64 are analyzed. Based on the above results, complexity measures of soft-decision maximum likelihood decoding for binary linear block codes are discussed.

2. Parallel Structure of Minimal Trellis Diagram

For simplicity, we only consider the trellis structure of binary linear block codes in this paper. The extension to nonbinary linear codes is straightforward.

Consider a binary linear (N, K) block code C. By an N-section trellis diagram for C[2],[3], we mean a directed graph, denoted T, such that (1) T has an initial state s_0 and a final state s_f (a state is simply a node in the graph), (2) each branch (an edge in the graph) has a label and two branches diverging from the same state have different labels, and (3) there is a directed path from s_0 to s_f with label sequence $u_1 u_2 \cdots u_N$ if and only if $(u_1, u_2, \ldots, u_N)$ is a codeword in C. In the following, a binary sequence of length ℓ is regarded as a binary ℓ-tuple, and vice versa. A trellis diagram for C with minimum number of states is said to be minimal, and a minimal trellis is unique within graph isomorphism[2],[3]. A subgraph of a trellis diagram is called a sub-trellis diagram.

For a nonnegative integer h not greater than N, let S_h denote the set of states of T just after the h-th bit position, where S_0 consists of the initial state s_0 only and S_N consists of the final state s_f only. Suppose that the code length N is a multiple of a positive integer L. An L-section minimal trellis diagram[3] for C can be obtained from the N-section minimal trellis diagram T by deleting every state in S_h for $h \in \{0, 1, \ldots, N\} - \{0, N/L, 2N/L, \ldots, N\}$ and every branch to or from a deleted state and by writing a branch with label α from a state $s \in S_{jN/L}$ to a state $s' \in S_{(j+1)N/L}$ for $0 \leq j < L$, if and only if there is a path with label α from s to s' in T. This L-section minimal trellis diagram may have parallel branches between two adjacent states with different labels.

Let h and h' be two integers such that $0 \leq h < h' \leq N$. For a binary N-tuple $\mathbf{v} = (v_1, v_2, \ldots, v_N)$, let $p_{h,h'}\mathbf{v}$ denote the binary $(h' - h)$-tuple $(v_{h+1}, v_{h+2}, \ldots, v_{h'})$ and let $p_{h,h'}[C]$ be defined as

$$p_{h,h'}[C] \stackrel{\triangle}{=} \{p_{h,h'}\mathbf{v} : \mathbf{v} \in C\}. \tag{1}$$

By the sub-trellis diagram of T for a subset U of $p_{0,h}[C]$, denoted $T[U]$, we mean the subgraph of T which consists of all states and branches on paths from the initial state to states in S_h whose labels belong to U. For such a sub-trellis diagram T', $p_{h,h'}[T']$ (or $p_{h,h'}T'$) denotes the sub-trellis diagram obtained by truncating T' except for the h-th to the h'-th bit positions.

Let $C_{h,h'}$ be the linear subcode of C consisting of all codewords whose components are all zero except for the $h' - h$ components from the $(h+1)$-th bit position to the h'-th bit

position. Let $K_{h,h'}$ be the dimension of $C_{h,h'}$, i.e.,

$$K_{h,h'} = \log_2 |C_{h,h'}|,$$

where for a set S, $|S|$ denotes the number of elements in S. For convenience, $K_{h,h}$ is defined as zero. For simplicity, we write $C^{tr}_{h,h'}$ for $p_{h,h'}[C_{h,h'}]$, the truncation of $C_{h,h'}$. $C^{tr}_{h,h'}$ is a linear subcode of $p_{h,h'}[C]$. Let $C_{\overline{h,h'}}$ denote the linear subcode of C consisting of all codewords whose components from the $(h+1)$-th bit position to the h'-th bit position are all zero, and let $K_{\overline{h,h'}}$ denote $\log_2 |C_{\overline{h,h'}}|$, the dimension of $C_{\overline{h,h'}}$. Note that for $0 \leq h < h' \leq N$,

$$|p_{h,h'}[C]| = 2^{K - K_{\overline{h,h'}}}. \tag{2}$$

For integers h, h' and h'' such that $0 \leq h < h' < h'' \leq N$, let $K_{h,h',h''}$ be defined as

$$K_{h,h',h''} \stackrel{\triangle}{=} K_{h,h''} - K_{h,h'} - K_{h',h''}. \tag{3}$$

For simplicity, we write K_h for $K_{0,h,N}$.

For two states s and s' in the minimal trellis diagram T of C, let $L(s, s')$ denote the set of all label sequences of paths from s to s'. For $\mathbf{u} \in p_{0,h}[C]$ with $0 < h \leq N$, let $\sigma\mathbf{u}$ (or $\sigma(\mathbf{u})$) denote the last state of the path labeled $\mathbf{u}$ from the initial state s_0 in T, and for a subset U of $p_{0,h}[C]$, $\sigma[U]$ denotes $\{\sigma\mathbf{u} : \mathbf{u} \in U\}$. For $\mathbf{u}_1$ and $\mathbf{u}_2 \in p_{0,h}[C]$, $\sigma\mathbf{u}_1 = \sigma\mathbf{u}_2$, if and only if

$$\mathbf{u}_1 + \mathbf{u}_2 \in C^{tr}_{0,h}. \tag{4}$$

For linear codes A and B such that $A \supseteq B$, let A/B denote the set of cosets in A with respect to B. For $\mathbf{u} \in A$, $A/B(\mathbf{u})$ denotes the coset of A/B to which $\mathbf{u}$ belongs. It follows from (4) that for $s \in S_h$,

$$L(s_0, s) \in p_{0,h}[C]/C^{tr}_{0,h}, \tag{5}$$

and it has been shown in [3, Appendix A] that

$$|S_h| = 2^{K_h}. \tag{6}$$

Let h be an integer such that $1 \leq h \leq N$, and let A and B be linear codes such that $C^{tr}_{0,h} \subseteq B \subseteq A \subseteq p_{0,h}[C]$. Let $\sigma[A/B]$ denote the following partition Π of $\{\sigma\mathbf{u}|\mathbf{u} \in A\}$: For two states s_1 and s_2 in $\{\sigma\mathbf{u}|\mathbf{u} \in A\}$, s_1 and s_2 are in the same block in Π, if and

only if there exist $\mathbf{u}_1$ and $\mathbf{u}_2$ in the same coset in A/B such that $s_1 = \sigma\mathbf{u}_1$ and $s_2 = \sigma\mathbf{u}_2$ [[1], equations (3) and (4); [3], Appendix A]. For $\mathbf{u}$ in a coset of A/B, let $\Pi(\mathbf{u})$ denote the block of Π: $\{\sigma(\mathbf{u}+\mathbf{v}) : \mathbf{v} \in B\}$. That is, there is the one-to-one correspondence between A/B and the set of blocks in Π. Each block in Π consists of $2^{K_B - K_{0,h}}$ states, where K_B is the dimension of B, and the number of blocks in Π, denoted $|\Pi|$, is given by

$$|\Pi| = 2^{K_A - K_B}, \tag{7}$$

where K_A denotes the dimension of A.

For two integers h and h' such that $1 \leq h < h' \leq N$, the structure of the truncated sub-trellis diagram $p_{h,h'}[T]$ of the minimal trellis diagram T for C has been investigated in [1]. To summarize the results, we first define the following notations. For two linear codes A and B of the same code length, let $A+B$ denote the linear code $\{\mathbf{u}+\mathbf{v} : \mathbf{u} \in A, \mathbf{v} \in B\}$. Let $\Gamma'_{h,h'}$ be defined as

$$\Gamma'_{h,h'} \stackrel{\triangle}{=} p_{0,h'}[C]/p_{0,h'}[C_{0,h'} + C_{h,N}]. \tag{8}$$

For the coset leader $\mathbf{u}$ of a coset in $\Gamma'_{h,h'}$, let $T_{0,h,h'}(\mathbf{u})$ denote the sub-trellis diagram of T for the coset $\Gamma'_{h,h'}(\mathbf{u})$ of $\Gamma'_{h,h'}$ with coset leader $\mathbf{u}$. Then, $T_{0,h,h'}(\mathbf{u})$ is obtained from $T_{0,h,h'}(\mathbf{0})$ simply by adding the j-th bit of $\mathbf{u}$ to the code bit label of each branch of $T_{0,h,h'}(\mathbf{0})$ at the j-th bit position for $1 \leq j \leq h'$.

Then, the set of states at the h'-th bit position in $T_{0,h,h'}(\mathbf{u})$ is $\sigma[\Gamma'_{h,h'}(\mathbf{u})]$. The block $\sigma[\Gamma'_{h,h'}(\mathbf{u})]$ consists of $2^{Q'_{h,h'}}$ states in $S_{h'}$, where

$$\begin{aligned} Q'_{h,h'} &= \log_2 |p_{0,h'}[C_{0,h'} + C_{h,N}]| - \log_2 |C^{tr}_{0,h'}| \\ &= K_{h,N} - K_{h,h'} - K_{h',N} \\ &= K_{h,h',N}. \end{aligned} \tag{9}$$

We write $T_{h,h'}(\mathbf{u})$ for $p_{h,h'}[T_{0,h,h'}(\mathbf{u})]$. Consider the set $p_{0,h}[\Gamma'_{h,h'}(\mathbf{u})] = \{p_{0,h}(\mathbf{u}+\mathbf{v}) : \mathbf{v} \in p_{0,h'}[C_{0,h'} + C_{h,N}]\}$. Since, $p_{0,h}[C_{0,h'} + C_{h,N}] = p_{0,h}[C_{0,h'}]$, we have that

$$p_{0,h}[\Gamma'_{h,h'}(\mathbf{u})] = \Gamma_{h,h'}(p_{0,h}\mathbf{u}), \tag{10}$$

where

$$\Gamma_{h,h'} = p_{0,h}[C]/p_{0,h}[C_{0,h'}]. \tag{11}$$

Then the set of states at the h-th bit position in $T_{h,h'}(\mathbf{u})$ is $\sigma[\Gamma_{h,h'}(p_{0,h}\mathbf{u})]$. The block $\sigma[\Gamma_{h,h'}(p_{0,h}\mathbf{u})]$ consists of $2^{Q_{h,h'}}$ states, where

$$\begin{aligned} Q_{h,h'} &\triangleq \log_2 |p_{0,h}[C_{0,h'}]| - \log_2 |C^{tr}_{0,h}| \\ &= K_{0,h'} - K_{h,h'} - K_{0,h} \\ &= K_{0,h,h'}. \end{aligned} \tag{12}$$

Then, $\log_2 |\sigma[\Gamma_{h,h'}]|$, denoted $q_{h,h'}$, is given by

$$\begin{aligned} q_{h,h'} &= K_h - Q_{h,h'} \\ &= K - K_{0,h'} + K_{h,h'} - K_{h,N}. \end{aligned} \tag{13}$$

Since $|S_{h'}| = 2^{K_{h'}}$, it follows from (9) and (13) that

$$\log_2 |\sigma[\Gamma'_{h,h'}]| = q_{h,h'}. \tag{14}$$

The following theorem (refer to Section 3 and Fig. 2 in [1] for the proof) describes the structure and connectivity of the N-section minimal trellis diagram T for a binary linear (N, K) code C.

Theorem 1: For $1 \leq h < h' \leq N$, let $\mathbf{u}_1 = \mathbf{0}, \mathbf{u}_2, \ldots, \mathbf{u}_{2^{q_{h,h'}}}$ be the coset leaders of $\Gamma'_{h,h'} \triangleq p_{0,h'}[C]/p_{0,h'}[C_{0,h'} + C_{h,N}]$, where $q_{h,h'} \triangleq K_h - K_{0,h,h'}$.
(1) The truncated trellis diagram $p_{h,h'}[T]$ of T consists of $2^{q_{h,h'}}$ structurally identical parallel sub-trellis diagrams $T_{h,h'}(\mathbf{u}_1), T_{h,h'}(\mathbf{u}_2), \ldots, T_{h,h'}(\mathbf{u}_{2^{q_{h,h'}}})$ without cross connections between them, where $T_{h,h'}(\mathbf{u}_i) \triangleq p_{h,h'}[T_{0,h,h'}(\mathbf{u}_i)]$ and $T_{0,h,h'}(\mathbf{u}_i)$ denotes the sub-trellis diagram of T for $\Gamma'_{h,h'}(\mathbf{u}_i)$.
(2) The set of states at the h-th bit position in $T_{h,h'}(\mathbf{u}_i)$ is $\sigma[\Gamma_{h,h'}(p_{0,h}\mathbf{u}_i)]$ and the set of states at the h'-th bit position in $T_{h,h'}(\mathbf{u}_i)$ is $\sigma[\Gamma'_{h,h'}(\mathbf{u}_i)]$, where $\Gamma_{h,h'}$ and $\Gamma'_{h,h'}$ are defined by (11) and (8), respectively. For every pair of states $s \in \sigma[\Gamma_{h,h'}(p_{0,h}\mathbf{u}_i)]$ and $s' \in \sigma[\Gamma'_{h,h'}(\mathbf{u}_i)]$, there are exactly $2^{K_{h,h'}}$ paths from s to s' in $T_{h,h'}(\mathbf{u}_i)$ and $L(s, s') \in p_{h,h'}[C]/C^{tr}_{h,h'}$.
(3) Sub-trellis diagram $T_{0,h,h'}(\mathbf{u}_i)$ is obtained from $T_{0,h,h'}(\mathbf{0})$ simply by adding the j-th bit of $\mathbf{u}_i$ to the code bit label of each branch of $T_{0,h,h'}(\mathbf{0})$ at the j-th bit position for $1 \leq j \leq h'$.

□

Next, we give three lemmas on the set of label sequences of paths between two states. First, since the sub-trellis diagram $T_{h,h'}(\mathbf{u}_i)$ is deterministic, we obtain Lemma 1.

Lemma 1: For a state s in $\sigma[\Gamma_{h,h'}(p_{0,h}\mathbf{u}_i)]$ and two states s'_1 and s'_2 in $\sigma[\Gamma'_{h,h'}(\mathbf{u}_i)]$,

$$L(s, s'_1) \cap L(s, s'_2) = \emptyset. \tag{15}$$

□

Let $\mathbf{u} = (u_1, u_2, \ldots, u_i)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_j)$ be two binary sequences of length i and j respectively. The concatenation of $\mathbf{u}$ and $\mathbf{v}$ is defined as the following sequence of length $i + j$:

$$\mathbf{u} \circ \mathbf{v} \stackrel{\triangle}{=} (u_1, u_2, \ldots, u_i, v_1, v_2, \ldots, v_j).$$

Let s_1 and s_2 be states in S_h, and let s'_1 and s'_2 be states in $S_{h'}$. Note that for $\mathbf{v}'_1 \in L(s_1, s'_1)$ and $\mathbf{v}'_2 \in L(s_2, s'_2)$,

$$\mathbf{v}'_1 = \mathbf{v}'_2, \tag{16}$$

if and only if there exist $\mathbf{v}_1 \in L(s_0, s_1)$ and $\mathbf{v}_2 \in L(s_0, s_2)$ such that

$$\mathbf{v}_1 \circ \mathbf{v}'_1 + \mathbf{v}_2 \circ \mathbf{v}'_2 = (\mathbf{v}_1 + \mathbf{v}_2) \circ \mathbf{0} \in p_{0,h'}[C_{\overline{h,h'}}]. \tag{17}$$

If $s'_1 = s'_2$ and $\mathbf{v}'_1 = \mathbf{v}'_2$, then $(\mathbf{v}_1 + \mathbf{v}_2) \circ \mathbf{0} \in C^{tr}_{0,h'}$ and therefore,

$$\mathbf{v}_1 + \mathbf{v}_2 \in C^{tr}_{0,h}. \tag{18}$$

Relations (4) and (18) imply the Lemma 2.

Lemma 2: For different states s_1 and s_2 in $\sigma[\Gamma_{h,h'}(p_{0,h}\mathbf{u}_i)]$ and a state s' in $\sigma[\Gamma'_{h,h'}(\mathbf{u}_i)]$,

$$L(s_1, s') \cap L(s_2, s') = \emptyset. \tag{19}$$

□

From (1) and (2) stated in Theorem 1, we have the following lemma.

Lemma 3: Let (s_1, s'_1) and (s_2, s'_2) be different state pairs in $S_h \times S_{h'}$. Suppose that

$$L(s_1, s'_1) \cap L(s_2, s'_2) \neq \emptyset. \tag{20}$$

Then the following equality holds:

$$L(s_1, s'_1) = L(s_2, s'_2). \tag{21}$$

□

Let $l_{h,h'}$ denote the number of different label sequences in the truncated trellis diagram $p_{h,h'}[T]$. Then, by (2)

$$l_{h,h'} = |p_{h,h'}[C]| = 2^{K-K_{\overline{h,h'}}}. \tag{22}$$

It follows from (2) in Theorem 1 and (22) that the number of different nonempty $L(s, s')$'s with $s \in S_h$ and $s' \in S_{h'}$, denoted $l'_{h,h'}$, is given by

$$l'_{h,h'} = l_{h,h'} 2^{-K_{h,h'}} = 2^{K-K_{\overline{h,h'}}-K_{h,h'}}. \tag{23}$$

3. Identical Parallel Components with the Same Branch Labels

In this section, we consider the label sequences of parallel sub-trellis diagrams $T_{h,h'}(\mathbf{u}_i)$ with $1 \le i \le 2^{q_{h,h'}}$, where $\mathbf{u}_1 = \mathbf{0}, \mathbf{u}_2, \ldots, \mathbf{u}_{2^{q_{h,h'}}}$ are the coset leaders of $\Gamma'_{h,h'} (= p_{0,h'}[C]/p_{0,h'}[C_{0,h'} + C_{h,N}])$. For $1 \le i \le 2^{q_{h,h'}}$, let $L_b(\mathbf{u}_i)$ denote the set of label sequences of $T_{h,h'}(\mathbf{u}_i)$, $\{p_{h,h'}(\mathbf{u}_i + \mathbf{v}) : \mathbf{v} \in p_{0,h'}[C_{0,h'} + C_{h,N}]\}$. We consider necessary and sufficient conditions under which

$$L_b(\mathbf{u}_i) = L_b(\mathbf{u}_j) \quad \text{for} \quad 1 \le i < j \le 2^{q_{h,h'}}. \tag{24}$$

The above equality (24) holds, if and only if

$$p_{h,h'}\mathbf{u}_i + p_{h,h'}\mathbf{u}_j \in p_{h,h'}[C_{0,h'} + C_{h,N}]. \tag{25}$$

Suppose that $L_b(\mathbf{u}_i)$ and $L_b(\mathbf{u}_j)$ have a common label sequence $\mathbf{v}$. Then, since $p_{h,h'}\mathbf{u}_i + \mathbf{v}$ and $p_{h,h'}\mathbf{u}_j + \mathbf{v}$ are in $p_{h,h'}[C_{0,h'} + C_{h,N}]$, the above relation (25) holds and we have the following lemma.

Lemma 4: Suppose that

$$L_b(\mathbf{u}_i) \cap L_b(\mathbf{u}_j) \neq \emptyset. \tag{26}$$

Then the equality (24) holds. □

The following lemma gives a necessary and sufficient condition for (24) in terms of $p_{0,h}\mathbf{u}_i$ and $p_{0,h}\mathbf{u}_j$.

Lemma 5: The equality (24) holds, if and only if

$$p_{0,h}\mathbf{u}_i + p_{0,h}\mathbf{u}_j \in p_{0,h}[C_{0,h'} + C_{\overline{h,h'}}]. \tag{27}$$

(Proof) Relation (25) implies that there is a binary h-tuple $\mathbf{u}$ such that

$$\mathbf{u} \circ p_{h,h'}(\mathbf{u}_i + \mathbf{u}_j) \in p_{0,h'}[C_{0,h'} + C_{h,N}], \tag{28}$$

and

$$\mathbf{u} \in p_{0,h}[C_{0,h'}]. \tag{29}$$

Since $\mathbf{u}_i + \mathbf{u}_j \in p_{0,h'}[C]$, it follows from (28) that $\mathbf{u}_i + \mathbf{u}_j + \mathbf{u} \circ p_{h,h'}(\mathbf{u}_i + \mathbf{u}_j) = (p_{0,h}(\mathbf{u}_i + \mathbf{u}_j) + \mathbf{u}) \circ \mathbf{0} \in p_{0,h'}[C]$, which implies

$$p_{0,h}(\mathbf{u}_i + \mathbf{u}_j) + \mathbf{u} \in p_{0,h}[C_{\overline{h,h'}}]. \tag{30}$$

Relation (27) follows from (29) and (30).

Conversely, assume (27) to be true for $\mathbf{u}_i$ and $\mathbf{u}_j$. Then, there exists $\mathbf{v}$ in $p_{h,h'}[C_{0,h'} + C_{\overline{h,h'}}]$ such that

$$p_{0,h}(\mathbf{u}_i + \mathbf{u}_j) \circ \mathbf{v} \in p_{0,h'}[C_{0,h'} + C_{\overline{h,h'}}]. \tag{31}$$

Since $\mathbf{u}_i + \mathbf{u}_j + p_{0,h}(\mathbf{u}_i + \mathbf{u}_j) \circ \mathbf{v} = \mathbf{0} \circ (p_{h,h'}(\mathbf{u}_i + \mathbf{u}_j) + \mathbf{v}) \in p_{0,h'}[C_{h,N}]$, it follows from (31) that $\mathbf{u}_i + \mathbf{u}_j \in p_{0,h'}[C_{0,h'} + C_{h,N} + C_{\overline{h,h'}}]$ and therefore relation (25) holds. □

Let $\Gamma_{\text{label},h,h'}$ denote $p_{0,h}[C]/p_{0,h}[C_{0,h'} + C_{\overline{h,h'}}]$. Then, from Lemma 5, equality (24) holds if and only if $p_{0,h}\mathbf{u}_i$ and $p_{0,h}\mathbf{u}_j$ belong to the same coset in $\Gamma_{\text{label},h,h'}$. Since $p_{0,h}[C_{0,h'} + C_{\overline{h,h'}}] \supseteq C_{0,h}^{tr}$, it follows from (4) that the equality (24) holds if and only if $\sigma(p_{0,h}\mathbf{u}_i)$ and $\sigma(p_{0,h}\mathbf{u}_j)$ are in the same block in $\sigma[\Gamma_{\text{label},h,h'}]$. Let $K''_{h,h'}$ be defined as

$$K''_{h,h'} \stackrel{\triangle}{=} \log_2 |p_{0,h}[C_{0,h'}] \cap p_{0,h}[C_{\overline{h,h'}}]|. \tag{32}$$

Then $\log_2 |\Gamma_{\text{label},h,h'}|$ is given by

$$\begin{aligned} \log_2 |\Gamma_{\text{label},h,h'}| &\stackrel{\triangle}{=} \log_2 |p_{0,h}[C]| - \log_2 |p_{0,h}[C_{0,h'} + C_{\overline{h,h'}}]| \\ &= K - K_{h,N} - (K_{0,h'} - K_{h,h'} + K_{\overline{h,h'}} - K_{h',N} - K''_{h,h'}) \\ &= q_{h,h'} - (K_{\overline{h,h'}} - K''_{h,h'} - K_{h',N}). \end{aligned} \tag{33}$$

Partition $\sigma[\Gamma_{h,h'}]$ is a refinement of partition $\sigma[\Gamma_{\text{label},h,h'}]$ and each block of $\sigma[\Gamma_{\text{label},h,h'}]$ consists of $2^{\lambda_{h,h'}}$ blocks of $\sigma[\Gamma_{h,h'}]$, where $\lambda_{h,h'}$ is given by

$$\lambda_{h,h'} \stackrel{\triangle}{=} K_{\overline{h,h'}} - K''_{h,h'} - K_{h',N}. \tag{34}$$

Then, Theorem 2 follows from Theorem 1, Lemmas 4 and 5, equations (33) and (34).

Theorem 2: Parallel sub-trellis diagrams $T_{h,h'}(\mathbf{u}_i)$ with $1 \leq i \leq 2^{q_{h,h'}}$ can be partitioned into $2^{q_{h,h'}-\lambda_{h,h'}}$ blocks of size $2^{\lambda_{h,h'}}$ in such a way that (1) if $T_{h,h'}(\mathbf{u}_i)$ and $T_{h,h'}(\mathbf{u}_j)$ are in the same block, they are identical up to the path labeling and (2) if there is a common label sequence in $T_{h,h'}(\mathbf{u}_i)$ and $T_{h,h'}(\mathbf{u}_j)$, they are in the same block. □

Example 1: For integers m and r with $0 \leq r \leq m$, let $\mathrm{RM}_{m,r}$ denote the r-th order Reed-Muller codes of length 2^m[5],[6]. For an example, consider the 16-section minimal trellis diagram for $\mathrm{RM}_{5,2}$. For the fourth section, $q_{8,10} = 4$, $Q_{8,10} = Q'_{8,10} = 1$ and $\lambda_{8,10} = 3$. That is, this section consists of sixteen 2-state parallel sub-trellis diagrams and these are partitioned into 2 blocks of size 8. For each block, sub-trellis diagrams in the block are identical up to the path labeling. The two representative sub-trellis diagrams are shown in Fig. 1.

The values of K_h, $K_{h,h'}$, $q_{h,h'}$, $Q_{h,h'}$, $Q'_{h,h'}$ and $\lambda_{h,h'}$ of the 2^i-section minimal trellis diagrams with $3 \leq i \leq 4$ for $\mathrm{RM}_{6,2}$ and $\mathrm{RM}_{6,3}$ are also listed in Tables 1.1, 1.2 and 2.1, 2.2, respectively. A trellis diagram T is said to be reversible if the graph T^R obtained from T by reversing the direction of each branch without changing the label and exchanging the initial state and the final state is identical to T. The minimal trellis diagrams for Reed-Muller codes, extended and permuted primitive BCH codes and their dual codes are reversible[4]. Therefore, the values are listed in Tables 1.1, 1.2, 2.1 and 2.2 only for the left-half of the trellis diagrams. □

4. The Detailed Structure of a Parallel Component

Next we consider the set of label sequences, $p_{h,h'}[C_{0,h'} + C_{h,N}]$, of the sub-trellis diagram $T_{h,h'}(\mathbf{0})$.

For simplicity, let A'_0, A'_1, A'_2, A'_3 and A'_4 denote the following linear codes related to C:

$$A'_0 \triangleq p_{0,h'}[C_{0,h'} + C_{h,N}], \tag{35}$$

$$A'_1 \triangleq p_{0,h'}[C_{0,h'}], \tag{36}$$

$$A'_2 \triangleq p_{0,h'}[C_{\overline{h,h'}} \cap (C_{0,h'} + C_{h,N})], \tag{37}$$

$$A'_3 \triangleq p_{0,h'}[C_{0,h}], \tag{38}$$

$$A'_4 \stackrel{\triangle}{=} p_{0,h'}[C_{h,h'}]. \tag{39}$$

Then, we have that

$$A'_1 \cap A'_2 = A'_3, \tag{40}$$

$$A'_1 + A'_2 \subseteq A'_0, \tag{41}$$

$$A'_4 \subseteq A'_1, \tag{42}$$

$$A'_2 \cap A'_4 = \{\mathbf{0}\}. \tag{43}$$

The dimensions of A'_0, A'_1, A'_3 and A'_4 are given by

$$\log_2 |A'_0| = Q'_{h,h'} + K_{0,h'}, \tag{44}$$

$$\log_2 |A'_1| = K_{0,h'}, \tag{45}$$

$$\log_2 |A'_3| = K_{0,h}, \tag{46}$$

$$\log_2 |A'_4| = K_{h,h'}. \tag{47}$$

Let $K'_{h,h'}$ denote the dimension of A'_2, that is,

$$K'_{h,h'} \stackrel{\triangle}{=} \log_2 |p_{0,h'}[C_{\overline{h,h'}} \cap (C_{0,h'} + C_{h,N})]|. \tag{48}$$

From (36) and (41), $\sigma[A'_0/(A'_1 + A'_2)]$ is well defined. Then, we have Lemma 6.

Lemma 6: (1) For $\mathbf{v}$ and $\mathbf{v}'$ in A'_0,

$$p_{h,h'}\mathbf{v} = p_{h,h'}\mathbf{v}', \tag{49}$$

if and only if

$$\mathbf{v} + \mathbf{v}' \in A'_2, \tag{50}$$

that is, $\mathbf{v}$ and $\mathbf{v}'$ are in the same coset in A'_0/A'_2.
(2) For $\mathbf{v}$ and $\mathbf{v}'$ in the same coset in A'_0/A'_2, (2.i) $\sigma\mathbf{v}$ and $\sigma\mathbf{v}'$ belong to the same block in $\sigma[A'_0/(A'_1 + A'_2)]$, and (2.ii) $\sigma\mathbf{v} = \sigma\mathbf{v}'$, if and only if $\mathbf{v}$ and $\mathbf{v}'$ are in the same coset in A'_0/A'_3.
(**Proof**) The part (1) follows from (17), and the part (2.i) follows from (4) and the definition of $\sigma[\cdot]$. If $\sigma\mathbf{v} = \sigma\mathbf{v}'$, then $\mathbf{v} + \mathbf{v}' \in A'_1$ from (4). Since $\mathbf{v} + \mathbf{v}' \in A'_2$ and $A'_1 \cap A'_2 = A'_3$, $\mathbf{v} + \mathbf{v}' \in A'_3$. Conversely, suppose that $\mathbf{v} + \mathbf{v}' \in A'_3$. Since $A'_3 \subseteq A'_1$, $\sigma\mathbf{v} = \sigma\mathbf{v}'$ from (4). □

Let A_0, A_2 and A_3 denote the following codes:

$$A_0 \stackrel{\triangle}{=} p_{0,h}[A'_0] = p_{0,h}[A'_1] = p_{0,h}[C_{0,h'}], \tag{51}$$

$$A_2 \stackrel{\triangle}{=} p_{0,h}[A'_2] = p_{0,h}[C_{\overline{h,h'}} \cap (C_{0,h'} + C_{h,N})], \tag{52}$$

$$A_3 \stackrel{\triangle}{=} p_{0,h}[A'_3] = p_{0,h}[C_{0,h}]. \tag{53}$$

Then we have that

$$A_3 \subseteq A_2 \subseteq A_0, \tag{54}$$

$$\log_2 |A_0| = K_{0,h'} - K_{h,h'}, \tag{55}$$

$$\log_2 |A_2| = \log_2 |A'_2| = K'_{h,h'}, \tag{56}$$

$$\log_2 |A_3| = K_{0,h}. \tag{57}$$

Let A'_{12} be defined as follows:

$$A'_{12} \stackrel{\triangle}{=} \{\mathbf{u} \in A'_1 : p_{0,h}\mathbf{u} \in A_2\}. \tag{58}$$

Then, $A'_3 + A'_4 \subseteq A'_{12} \subseteq A'_1$. This relation and (40) imply that

$$A'_{12} \cap A'_2 = A'_3. \tag{59}$$

For $\mathbf{u}$ and $\mathbf{u}'$ in A'_{12} such that $p_{0,h}\mathbf{u} = p_{0,h}\mathbf{u}'$, $\mathbf{u} + \mathbf{u}' = \mathbf{0} \circ p_{h,h'}(\mathbf{u} + \mathbf{u}') \in p_{0,h'}[C_{h,h'}] = A'_4$. Since $p_{0,h}[A'_1] = A_0 \supseteq A_2$, $p_{0,h}[A'_{12}] = A_2$. Consequently,

$$\begin{aligned} \log_2 |A'_{12}| &= \log_2 |A_2| + \log_2 |A'_4| \\ &= K'_{h,h'} + K_{h,h'}. \end{aligned} \tag{60}$$

From (54), $\sigma[A_0/A_2]$ is well defined. The next lemma holds.

Lemma 7: Suppose that $\mathbf{v}$ and $\mathbf{v}'$ are in the same coset in $A'_0/(A'_1 + A'_2)$. Then, (i) if and only if $\mathbf{v}$ and $\mathbf{v}'$ are in the same coset in $A'_0/(A'_2 + A'_{12})$, then $\sigma(p_{0,h}\mathbf{v})$ and $\sigma(p_{0,h}\mathbf{v}')$ are in the same block in $\sigma[A_0/A_2]$, and (ii) if and only if $\mathbf{v}$ and $\mathbf{v}'$ are in the same coset in $A'_0/(A'_3 + A'_4)$, then $\sigma(p_{0,h}\mathbf{v}) = \sigma(p_{0,h}\mathbf{v}')$.

(**Proof**) Since $p_{0,h}[A'_2 + A'_{12}] = A_2$, the if-part of (i) is true. If $\sigma(p_{0,h}\mathbf{v})$ and $\sigma(p_{0,h}\mathbf{v}')$ are in the same block in $\sigma[A_0/A_2]$, then $p_{0,h}(\mathbf{v} + \mathbf{v}') \in A_2$. Since $\mathbf{v} + \mathbf{v}' \in A'_1 + A'_2$, $\mathbf{v} + \mathbf{v}'$

can be expressed as $\mathbf{v}+\mathbf{v}' = \mathbf{u}+\mathbf{u}'$ where $\mathbf{u} \in A'_1$ and $\mathbf{u}' \in A'_2$. Since $p_{0,h}\mathbf{u}' \in A_2$ and $p_{0,h}(\mathbf{u}+\mathbf{u}') \in A_2$, $p_{0,h}\mathbf{u} \in A_2$ and therefore, $\mathbf{u} \in A'_{12}$.

Consider the part (ii). If $\mathbf{v}$ and $\mathbf{v}'$ are in the same coset in $A'_0/(A'_3+A'_4)$, then $\mathbf{v}+\mathbf{v}' \in p_{0,h'}[C_{0,h}+C_{h,h'}]$ and therefore, $p_{0,h}\mathbf{v}+p_{0,h}\mathbf{v}' \in p_{0,h}[C_{0,h}]$, which implies that $\sigma(p_{0,h}\mathbf{v}) = \sigma(p_{0,h}\mathbf{v}')$ by (4). Conversely, assume that $\sigma(p_{0,h}\mathbf{v}) = \sigma(p_{0,h}\mathbf{v}')$. From (4), $p_{0,h}(\mathbf{v}+\mathbf{v}') \in C^{tr}_{0,h}$. Hence, $\mathbf{v}+\mathbf{v}'+p_{0,h}(\mathbf{v}+\mathbf{v}')\circ\mathbf{0} = \mathbf{0}\circ p_{h,h'}(\mathbf{v}+\mathbf{v}') \in (A'_1+A'_2)\cap p_{0,h'}[C_{h,N}] = p_{0,h'}[C_{h,h'}]$. Consequently, $\mathbf{v}+\mathbf{v}' = p_{0,h}(\mathbf{v}+\mathbf{v}')\circ\mathbf{0}+\mathbf{0}\circ p_{h,h'}(\mathbf{v}+\mathbf{v}') \in A'_3+A'_4$. □

The dimensions of $A'_1+A'_2$, $A'_{12}+A'_2$ and $A'_2+A'_4$ are given by

$$\begin{aligned}
\log_2|A'_1+A'_2| &= K_{0,h'}+K'_{h,h'}-K_{0,h} \\
&\quad \text{(by (40), (45), (46) and (48))}, && (61)\\
\log_2|A'_{12}+A'_2| &= 2K'_{h,h'}+K_{h,h'}-K_{0,h} \\
&\quad \text{(by (46), (48), (59) and (60))}, && (62)\\
\log_2|A'_2+A'_4| &= K'_{h,h'}+K_{h,h'} \\
&\quad \text{(by (43), (47) and (48))}. && (63)
\end{aligned}$$

Let U be a coset of A'_0/A'_2. Since $A'_4 \subseteq A'_{12} \subseteq A'_1$, there exist uniquely a coset U_4 of $A'_0/(A'_2+A'_4)$, a coset U_{12} of $A'_0/(A'_{12}+A'_2)$ and a coset U_1 of $A'_0/(A'_1+A'_2)$ such that

$$U \subseteq U_4 \subseteq U_{12} \subseteq U_1. \tag{64}$$

Consider the following sub-trellis diagrams of $T_{h,h'}(\mathbf{0})$.

(i) $p_{h,h'}T[U]$:

(i.1) From (1) in Lemma 6, $p_{h,h'}[U]$ consists of a single $(h'-h)$-tuple, denoted $\mathbf{u}$. U consists of $2^{K'_{h,h'}-K_{0,h}}$ cosets of A'_0/A'_3 and intersects each of those $2^{K'_{h,h'}-K_{0,h}}$ cosets of A'_0/A'_1 in the coset U_1 of $A'_0/(A'_1+A'_2)$ at a coset of A'_0/A'_3 (by (40), (46) and (48)). From (4) and (61), each block of $\sigma[A'_0/(A'_1+A'_2)]$ consists of $2^{K'_{h,h'}-K_{0,h}}$ states.

(i.2) It follows from the above (i.1) and (2) in Lemma 6 that $\sigma[U]$, the set of states at the h'-th bit position in $p_{h,h'}T[U]$, forms exactly a block of $\sigma[A'_0/(A'_1+A'_2)]$, denoted B', consisting of $2^{K'_{h,h'}-K_{0,h}}$ states, and B' depends on U_1 only.

(i.3) Since a block of $\sigma[A_0/A_2]$ consists of $2^{K'_{h,h'}-K_{0,h}}$ states (by (4), (56) and (57)), Lemma 7 implies that $\sigma[p_{0,h}[U]]$, the set of states at the h-th bit position in $p_{h,h'}T[U]$, forms exactly

a block of $\sigma[A_0/A_2]$, denoted B, which depends on U_{12} only. From Lemmas 1, 2 and 6, there exists a one-to-one mapping φ from B to B' such that there is a path labeled with $\mathbf{u}$ from each state s in B to the corresponding state $\varphi(s)$ in B' and there are no other paths in $p_{h,h'}T[U]$. From (ii) in Lemma 7, φ depends on U_4 only.

(ii) $p_{h,h'}T[U_4]$:

(ii.1) U_4 consists of $2^{K_{h,h'}}$ cosets, denoted $U_{2,1}, \ldots, U_{2,2^{K_{h,h'}}}$, of A'_0/A'_2 (by (48) and (63)), and let $p_{h,h'}[U_{2,i}] \stackrel{\triangle}{=} \{\mathbf{u}_i\}$ for $1 \le i \le 2^{K_{h,h'}}$. Then, $p_{h,h'}[U_4] = \{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{2^{K_{h,h'}}}\}$ forms a coset of $p_{h,h'}[A'_0]/p_{h,h'}[A'_4]$.

(ii.2) Since B, B' and the mapping φ defined in (i) are the same for each coset of A'_0/A'_2 in U_4, it follows from (2) in Theorem 1 that $\sigma[p_{0,h}[U_4]] = B$, that $\sigma[U_4] = B'$ and for all s in B, $L(s, \varphi(s))$ is the same coset $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_{2^{K_{h,h'}}}\}$ of $p_{h,h'}[A'_0]/p_{h,h'}[A'_4]$, and $L(s, s') = \emptyset$ for $s' \neq \varphi(s)$.

(iii) $p_{h,h'}T[U_{12}]$:

U_{12} consists of $2^{K'_{h,h'}-K_{0,h}}$ cosets, denoted $U_{4,1}, \ldots, U_{4,2^{K'_{h,h'}-K_{0,h}}}$, of $A'_0/(A'_2+A'_4)$ (by (62) and (63)). For $1 \le i \le 2^{K'_{h,h'}-K_{0,h}}$, $\sigma[p_{0,h}[U_{4,i}]] = B$, $\sigma[U_{4,i}] = B'$ and let φ_i denote the one-to-one mapping from B to B' such that there exist paths from $s \in B$ and $s' \in B'$ in $p_{h,h'}T[U_{4,i}]$ if and only if $s' = \varphi_i(s)$. It follows from (2) in Theorem 1 that for $1 \le i < j \le 2^{K'_{h,h'}-K_{0,h}}$,

$$\varphi_i(s) \neq \varphi_j(s), \quad \text{for} \quad s \in B,$$
$$\varphi_i^{-1}(s') \neq \varphi_j^{-1}(s'), \quad \text{for} \quad s' \in B',$$

where φ_i^{-1} denotes the reverse mapping of φ_i.

(iv) $p_{h,h'}T[U_1]$:

U_1 consists of $2^{K_{0,h'}-K'_{h,h'}-K_{h,h'}}$ cosets of $A'_0/(A'_{12}+A'_2)$, denoted $U_{12,i}$ with $1 \le i \le 2^{K_{0,h'}-K'_{h,h'}-K_{h,h'}}$ (by (61) and (62)). For all cosets $U_{12,i}$, $\sigma[U_{12,i}]$ are the same B'. Since $p_{0,h}[U_1] = A_0$ (by (51)), $\sigma[p_{0,h}[U_{12,i}]]$ with $1 \le i \le 2^{K_{0,h'}-K'_{h,h'}-K_{h,h'}}$ run on all $2^{K_{0,h'}-K'_{h,h'}-K_{h,h'}}$ blocks of $\sigma[A_0/A_2]$.

The following theorem summarizes the above results.

Theorem 3: Let Γ and Γ' denote $A_0/A_2 = p_{0,h}[C_{0,h'}]/p_{0,h}[C_{\overline{h,h'}} \cap (C_{0,h'} + C_{h,N})]$ and $A'_0/(A'_1 + A'_2) = p_{0,h'}[C_{0,h'} + C_{h,N}]/p_{0,h'}[C_{0,h'} + C_{\overline{h,h'}} \cap (C_{0,h'} + C_{h,N})]$, respectively. In sub-trellis diagram $T_{h,h'}(\mathbf{0})$, for each pair of blocks $B \in \sigma[\Gamma]$ and $B' \in \sigma[\Gamma']$ consisting of

$2^{K'_{h,h'}-K_{0,h}}$ states respectively, the set of paths from states in B to states in B' can be partitioned into $2^{K_{h,h'}-K_{0,h}}$ subsets $P_1, P_2, \ldots, P_{2^{K'_{h,h'}-K_{0,h}}}$ in the following way:
(1) For $1 \leq i \leq 2^{K'_{h,h'}-K_{0,h}}$, there is a one-to-one correspondence φ_i between B and B' such that for every state s in B and $1 \leq i < j \leq 2^{K'_{h,h'}-K_{0,h}}$, $\varphi_i(s) \neq \varphi_j(s)$,
(2) from each state s in B to the corresponding state $\varphi_i(s)$ in B', there are $2^{K_{h,h'}}$ parallel paths in P_i whose set of label sequences is the same coset of $p_{h,h'}[A'_0]/p_{h,h'}[A'_1 + A'_2] = p_{h,h'}[A'_0]/C^{tr}_{h,h'}$ for every s in B, and
(3) there are no other paths in P_i.

In $T_{h,h'}(\mathbf{0})$, there are no paths with a same label sequence other than those stated above. □

Example 2: Consider the 4-section minimal trellis diagram for $\mathrm{RM}_{4,2}$. The second section consists of two 4-state parallel sub-trellis diagrams. In sub-trellis diagram $T_{4,8}(\mathbf{0})$, since $K'_{4,8} - K_{0,4} = 2$, there is only one block, denoted B, in $\sigma[\Gamma]$ (or denoted B', in $\sigma[\Gamma']$) consisting of four states. The set of paths from states in B to states in B' can be partitioned into four subsets P_1, P_2, P_3, P_4. A one-to-one correspondence φ_i with $1 \leq i \leq 4$ is shown in Fig.2 (a). The detailed structure of the parallel component, $T_{4,8}(\mathbf{0})$, is depicted in Fig.2 (b).

The values of $K'_{h,h'} - K_{0,h}$ of the 2^i-section minimal trellis diagrams with $3 \leq i \leq 4$ for $\mathrm{RM}_{6,2}$ and $\mathrm{RM}_{6,3}$ are also listed in Tables 1.1, 1.2 and 2.1, 2.2, respectively. □

5. Complexity Measures of Soft-Decision Maximum Likelihood Decoding

We consider the complexity of the maximum likelihood decoding (Viterbi decoding) for linear codes of length N when an L-section minimal trellis diagram is used. The complexity is measured by (M1) the total number BM of the branch metric computations, (M2) the total number CPP of comparisons to find the smallest branch metric among each set of parallel branches, (M3) the total number AD of additions of the smallest branch metric among those parallel branches and the survivor's metric of the state from which the branches diverge, and (M4) the total number CP of comparisons to find a survivor at each state[7].

For $1 \leq i \leq L$, let h_i and h'_i denote $(i-1)N/L$ and iN/L, respectively. As stated in Theorems 1 and 2, the i-th section of the trellis diagram for $1 \leq i \leq L$ consists of $2^{q_{h_i,h'_i}}$

structurally identical parallel components without cross connections between them which are partitioned into $2^{q_{h_i,h_i'}-\lambda_{h_i,h_i'}}$ blocks, called identical parallel component blocks, each of which consists of $2^{\lambda_{h_i,h_i'}}$ parallel components identical up to the path labeling. From Lemma 4, the branch metric computations and comparisons among parallel branches can be performed separately in each identical parallel component blocks without duplication. Let BM_i and CPP_i denote the numbers of the branch metric computations and comparisons among parallel branches in an identical parallel component block in the i-th section, respectively. The numbers BM_i, BM, CPP_i and CPP are given as follows:

$$
\begin{aligned}
BM_i &= 2^{K-K_{\overline{h_i,h_i'}}-q_{h_i,h_i'}+\lambda_{h_i,h_i'}} \quad \text{(by (22))}, && (65)\\
BM &= \sum_{i=1}^{L} BM_i 2^{q_{h_i,h_i'}-\lambda_{h_i,h_i'}} \\
&= \sum_{i=1}^{L} 2^{K-K_{\overline{h_i,h_i'}}}, && (66)\\
CPP_i &= (2^{K_{h_i,h_i'}}-1)2^{K-K_{\overline{h_i,h_i'}}-K_{h_i,h_i'}-q_{h_i,h_i'}+\lambda_{h_i,h_i'}} \quad \text{(by (23))}, && (67)\\
CPP &= \sum_{i=1}^{L} CPP_i 2^{q_{h_i,h_i'}-\lambda_{h_i,h_i'}} \\
&= \sum_{i=1}^{L} (1-2^{-K_{h_i,h_i'}})2^{K-K_{\overline{h_i,h_i'}}}. && (68)
\end{aligned}
$$

Let AD_i' and CP_i' be the numbers of additions and comparisons to find the survivor's metric of a state, respectively, performed for each parallel component $T_{h_i,h_i'}(\mathbf{u})$ in the i-th section. Conventional evaluations on AD_i', AD, CP_i' and CP are given by

$$
\begin{aligned}
AD_i' &= \begin{cases} 2^{Q_{h_i,h_i'}+Q'_{h_i,h_i'}}, & \text{for} \quad 2 \le i \le L, \\ 0, & \text{for} \quad i=1, \end{cases} && (69)\\
AD &= \sum_{i=1}^{L} AD_i' 2^{q_{h_i,h_i'}} \\
&= \sum_{i=2}^{L} 2^{Q_{h_i,h_i'}+K_{h_i'}} = \sum_{i=2}^{L} 2^{K_{h_i}+Q'_{h_i,h_i'}}, && (70)\\
CP_i' &= \begin{cases} (2^{Q_{h_i,h_i'}}-1)2^{Q'_{h_i,h_i'}}, & \text{for} \quad 2 \le i \le L, \\ 0, & \text{for} \quad i=1, \end{cases} && (71)\\
CP &= \sum_{i=1}^{L} CP_i' 2^{q_{h_i,h_i'}}
\end{aligned}
$$

$$= \sum_{i=2}^{L} (2^{Q_{h_i,h_i'}} - 1) 2^{K_{h_i'}}. \tag{72}$$

In Tables 1.1, 1.2, 2.1 and 2.2, the evaluations of BM_i, CPP_i, AD_i' and CP_i' for $\mathrm{RM}_{6,2}$ and $\mathrm{RM}_{6,3}$ are also listed. Note that $BM_i = BM_{L+1-i}$, $CCP_i = CCP_{L+1-i}$ when the trellis diagram is reversible. The numbers of additions and comparisons in each sub-trellis diagram $p_{h_i,h_i'}T[U_{12}]$ stated in Section 4 can be reduced by an algorithm presented in Appendix A. The values of BM, CPP, AD and CP for the 2^i-section minimal trellis diagrams with $1 \leq i \leq 6$ for $\mathrm{RM}_{6,2}$ and $\mathrm{RM}_{6,3}$ are listed in Tables 1.3 and 2.3, respectively.

Next, we consider the space complexity. The space to store the trellis diagram may be reduced by using the fact that each section consists of identical parallel component blocks. The entire L-section minimal trellis diagram for C consists of the first tree type section, $2^{q_{N/L,(L-1)N/L}}$ structurally identical (except branch labels) sub-trellis diagrams without cross connection between them, and the last tree type section. For example, the 4-section trellis diagram for $\mathrm{RM}_{6,2}$ consists of the first tree type section, 64 structurally identical 16-state sub-trellis diagrams without cross connection between them, and the last tree type section. By finding the survivor path for each sub-trellis diagram sequentially, we can reduce the memory to store the survivor path and the survivor's metric at states.

6. Conclusion

In this paper, we have expressed the number of identical parallel subgraphs with the same branch labels in each section of an L-section minimal trellis diagram for a linear block code in terms of the dimensions of specific linear codes related to the given code. We have also shown that each parallel component has simple structure. These results may be used effectively in implementing Viterbi decoding.

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Appendix A

A decoding algorithm for sub-trellis diagram $p_{h,h'}T[U_{12}]$: The same notations as in Section 4 are used. For simplicity, parallel component $T_{h,h'}(\mathbf{0})$ is considered, and let IPCB denote the block consisting of all parallel components identical with $T_{h,h'}(\mathbf{0})$ up to the path labeling. Decoding for a parallel component is the same as that for another component in IPCB except that the survivor's metric at a state in a component is probably different from the one at the corresponding state in the other component.

Suppose that branch metrics in IPCB are computed and the smallest branch metrics among parallel branches are found and can be retrieved in the manner stated below.

Let U' be a coset of $A'_0/(A'_1 + A'_2)$. For state blocks $B \in \sigma[p_{0,h}[U']/A_2] = \sigma[A_0/A_2]$ and $B' \in \sigma[U']$, number the states in B and B', respectively, as follows:

$$B = \{s_1, s_2, \cdots, s_\nu\}, \tag{A·1}$$

$$B' = \{s'_1, s'_2, \cdots, s'_\nu\}, \tag{A·2}$$

where $\nu \stackrel{\triangle}{=} 2^{K'_{h,h'} - K_{0,h}}$. For $1 \le i \le \nu$, the set of label sequences in path set P_i (stated in Theorem 3) is that of a set of parallel branches. Let $m(B, B', i)$ denote the smallest of the metrics of the parallel path label sequences for P_i.

Let π' be a permutation on $\{1, 2, \cdots, \nu\}$ such that

$$m(B, B', \pi'(1)) \le m(B, B', \pi'(2)) \le \cdots \le m(B, B', \pi'(\nu)). \tag{A·3}$$

This π' can be found by a sorting operation. This sorting is to be done only once for each ordered pair of state blocks corresponding to B and B' in all parallel components in IPCB. For $1 \le i \le \nu$, let $sm(i)$ denote the survivor's metric at state s_i. It follows from Theorem 3 that for $1 \le j \le \nu$, the smallest metric, denoted $sm'(B, j)$, at state s'_j when the metrics of paths from states in B to s'_j are only considered is given by

$$sm'(B, j) = \min_{1 \le i \le \nu,\ s_j = \varphi_r(s_i)} sm(i) + m(B, B', r). \tag{A·4}$$

Then, the survivor's metric at state s'_j, denoted $sm'(j)$, is given by

$$sm'(j) = \min_{B \in \sigma[A_0/A_2]} sm'(B, j). \tag{A·5}$$

Now, we present a procedure for finding $\{sm'(B,1), sm'(B,2), \cdots, sm'(B,\nu)\}$.

1) By sorting $\{sm(1), sm(2), \cdots, sm(\nu)\}$, find a permutation π on $\{1, 2, \cdots, \nu\}$ such that

$$sm(\pi(1)) \leq sm(\pi(2)) \leq \cdots \leq sm(\pi(\nu)). \tag{A·6}$$

2) For $1 \leq i \leq \nu$ and $1 \leq r \leq \nu$, let $\varphi'_r(i) \stackrel{\triangle}{=} j$, where j satisfies $s'_j = \varphi_{\pi'(r)}(s_{\pi(i)})$ and $m'(i,r) \stackrel{\triangle}{=} sm(\pi(i)) + m(B, B', \pi'(r))$. Then, from (A·3), (A·4) and (A·6),

$$m'(i,r) \leq m'(i',r'), \quad \text{for } 1 \leq i \leq i' \leq \nu \text{ and } 1 \leq r \leq r' \leq \nu, \tag{A·7}$$

$$sm'(B,j) = \min_{1 \leq i \leq \nu,\ j = \varphi'_r(i)} m'(i,r). \tag{A·8}$$

For $1 \leq j \leq \nu$, let $rm(j)$ be the smallest r for which there is a positive integer i less than $\nu + 1$ such that $j = \varphi'_{r-i+1}(i)$. Since $\{\varphi'_r(1) : 1 \leq r \leq \nu\} = \{1, 2, \cdots, \nu\}$,

$$rm(j) \leq \nu, \quad \text{for } 1 \leq j \leq \nu. \tag{A·9}$$

It follows from (A·7) to (A·9) that for $1 \leq j \leq \nu$,

$$sm'(B,j) = \min_{i \in I_j} m'(i, rm(j) - i + 1), \tag{A·10}$$

where

$$I_j \stackrel{\triangle}{=} \{i : 1 \leq i \leq rm(j), j = \varphi'_{rm(j)-i+1}(i)\}. \tag{A·11}$$

The number of additions for obtaining $m'(i, rm(j) - i + 1)$ in the right-hand side of (A·10) from $sm(\pi(i))$ and $m(B, B', \pi'(r))$ is $|I_j|$ and the number of comparisons to find the smallest among $\{m'(i, rm(j) - i + 1) : i \in I_j\}$ is $|I_j| - 1$. Hence the total number, denoted $a(B)$, of the above additions and that, denoted $c(B)$, of the above comparisons for finding $\{sm'(B,1), sm'(B,2), \cdots, sm'(B,\nu)\}$ are given by

$$a(B) = \sum_{j=1}^{\nu} |I_j|, \tag{A·12}$$

$$c(B) = \sum_{j=1}^{\nu} (|I_j| - 1) = a(B) - \nu. \tag{A·13}$$

Note that

$$\sum_{j \in \{j : rm(j) = r\}} |I_j| \leq r. \tag{A·14}$$

It follows from (A·9) and (A·12) to (A·14) that

$$a(B) \leq \nu(\nu+1)/2, \tag{A·15}$$

$$c(B) \leq \nu(\nu-1)/2. \tag{A·16}$$

For the worst case, the equalities in (A·15) and (A·16) hold. However, the upper bounds are not tight in general, and the average values of $a(B)$ and $c(B)$ are much smaller.

Since the number of blocks in $\sigma[A_0/A_2]$ and $\sigma[A'_0/(A'_1+A'_2)]$ are $2^{Q_{h,h'}}/\nu$ and $2^{Q'_{h,h'}}/\nu$, respectively, the total numbers $AD''_{h,h'}$ and $CP''_{h,h'}$ of the above additions and comparisons for finding the survivor's metrics at all states at the h'-th bit position of $T_{h,h'}(\mathbf{0})$ are upper bounded as follows:

$$AD''_{h,h'} \leq \frac{\nu+1}{2\nu}2^{Q_{h,h'}+Q'_{h,h'}}, \tag{A·17}$$

$$CP''_{h,h'} \leq \frac{\nu-1}{2\nu}2^{Q_{h,h'}+Q'_{h,h'}} + \left(\frac{2^{Q_{h,h'}}}{\nu}-1\right)2^{Q'_{h,h'}} + const.\cdot 2^{Q_{h,h'}}\log_2\nu, \tag{A·18}$$

$$\leq \left(\frac{\nu+1}{2\nu}2^{Q_{h,h'}}-1\right)2^{Q'_{h,h'}} + const.\cdot 2^{Q_{h,h'}}\log_2\nu, \tag{A·19}$$

where $\nu = 2^{K'_{h,h'}-K_{0,h}}$.

The second and third terms in the right-hand side of (A·18) are the numbers of comparisons to find the minimum in the right-hand side of (A·5) and the permutation π for every block in $\sigma[A_0/A_2]$. The following number of comparisons to find the permutation π' for every ordered pair of blocks in $\sigma[A_0/A_2]$ and $\sigma[A'_0/(A'_1+A'_2)]$ must be taken into account for each identical parallel component block:

$$const.\cdot\frac{2^{Q_{h,h'}+Q'_{h,h'}}}{\nu}\log_2\nu. \tag{A·20}$$

From (69), (71), (A·17) and (A·19), we see that the improving factors, $AD''_{(i-1)N/L,iN/L}/AD'_i$ and $CP''_{(i-1)N/L,iN/L}/CP'_i$ are about $(\nu+1)/2\nu$ by the worst case analysis. The factor will be considerably improved by the average analysis.

Table 1.1: The measures of structural complexity of parallel components of the 8-section minimal trellis diagram for $\mathrm{RM}_{6,2}$.

i	1	2	3	4
$K_{8(i-1)}$	0	7	10	13
K_{8i}	7	10	13	10
$K_{8(i-1),8i}$	0			
$q_{8(i-1),8i}$	0	6	9	9
$Q_{8(i-1),8i}$	0	1	1	4
$Q'_{8(i-1),8i}$	7	4	4	1
$\lambda_{8(i-1),8i}$	0	3	6	6
$K'_{8(i-1),8i} - K_{0,8(i-1)}$	0	1	1	1
BM_i	128	16	16	16
CPP_i	0			
AD'_i	0	32	32	32
AD'_{9-i}	128	32	32	32
CP'_i	0	16	16	30
CP'_{9-i}	127	30	30	16

(1) The number of states at the end of the i-th section (or just after the $8i$-th bit) is $2^{K_{8i}}$.
(2) For each state s at the $8i$-th bit, there are $2^{Q_{8(i-1),8i}}$ states at the $8(i-1)$-th bit from which there are branches to s, and the number of parallel branches is $2^{K_{8(i-1),8i}}$.
(3) The number of parallel components in the i-th section is $2^{q_{8(i-1),8i}}$.
(4) The number of initial states of a parallel component in the i-th section is $2^{Q_{8(i-1),8i}}$.
(5) The number of final states of a parallel component in the i-th section is $2^{Q'_{8(i-1),8i}}$.
(6) The $2^{q_{8(i-1),8i}}$ parallel components in the i-th section are partitioned into $2^{q_{8(i-1),8i}-\lambda_{8(i-1),8i}}$ blocks of size $2^{\lambda_{8(i-1),8i}}$ in such a way that two parallel components are identical up to branch labels if and only if they are in a same block.
(7) Each parallel component in the i-th section has the structure stated in Theorem 3.
(8) The complexity measures of soft-decision maximum likelihood decoding is discussed in Section 5.

Table 1.2: The measures of structural complexity of parallel components of the 16-section minimal trellis diagram for $\mathrm{RM}_{6,2}$.

i	1	2	3	4	5	6	7	8
$K_{4(i-1)}$	0	4	7	10	10	13	13	13
K_{4i}	4	7	10	10	13	13	13	10
$K_{4(i-1),4i}$	0							
$q_{4(i-1),4i}$	0	4	7	9	10	12	12	10
$Q_{4(i-1),4i}$	0	0	0	1	0	1	1	3
$Q'_{4(i-1),4i}$	4	3	3	1	3	1	1	0
$\lambda_{4(i-1),4i}$	0	3	6	6	9	9	9	9
$K'_{4(i-1),4i} - K_{0,4(i-1)}$	0	0	0	1	0	1	1	0
BM_i	16	8	8	2	8	2	2	8
CPP_i	0							
AD'_i	0	8	8	4	8	4	4	8
AD'_{17-i}	16	8	8	4	8	4	4	8
CP'_i	0	0	0	2	0	2	2	7
CP'_{17-i}	15	7	7	2	7	2	2	0

Table 1.3: The complexity measures of soft-decision maximum likelihood decoding for $\mathrm{RM}_{6,2}$ when an L-section minimal trellis diagram is used.

L	2	4	8	16	32	64
BM	131072	8192	1024	256	128	128
CPP	129024	4096	0	0	0	0
AD	2048	33792	69760	104720	191908	375034
CP	1023	31743	50175	50175	50175	50175

Table 2.1: The measures of structural complexity of parallel components of the 8-section minimal trellis diagram for $\mathrm{RM}_{6,3}$.

i	1	2	3	4
$K_{8(i-1)}$	0	7	10	13
K_{8i}	7	10	13	10
$K_{8(i-1),8i}$	1			
$q_{8(i-1),8i}$	0	4	7	7
$Q_{8(i-1),8i}$	0	3	3	6
$Q'_{8(i-1),8i}$	7	6	6	3
$\lambda_{8(i-1),8i}$	0	3	6	6
$K'_{8(i-1),8i} - K_{0,8(i-1)}$	0	3	3	3
BM_i	256	128	128	128
CPP_i	128	64	64	64
AD'_i	0	512	512	512
AD'_{9-i}	128	512	512	512
CP'_i	0	448	448	504
CP'_{9-i}	127	504	504	448

Table 2.2: The measures of structural complexity of parallel components of the 16-section minimal trellis diagram for $\mathrm{RM}_{6,3}$.

i	1	2	3	4	5	6	7	8
$K_{4(i-1)}$	0	4	7	10	10	13	13	13
K_{4i}	4	7	10	10	13	13	13	10
$K_{4(i-1),4i}$	0							
$q_{4(i-1),4i}$	0	3	6	7	9	10	10	9
$Q_{4(i-1),4i}$	0	1	1	3	1	3	3	4
$Q'_{4(i-1),4i}$	4	4	4	3	4	3	3	1
$\lambda_{4(i-1),4i}$	0	3	6	6	9	9	9	9
$K'_{4(i-1),4i} - K_{0,4(i-1)}$	0	1	1	3	1	3	3	1
BM_i	16	16	16	8	16	8	8	16
CPP_i	0							
AD'_i	0	32	32	64	32	64	64	32
AD'_{17-i}	16	32	32	64	32	64	64	32
CP'_i	0	16	16	56	16	56	56	30
CP'_{17-i}	15	30	30	56	30	56	56	16

Table 2.3: The complexity measures of soft-decision maximum likelihood decoding for $\mathrm{RM}_{6,3}$ when an L-section minimal trellis diagram is used.

L	2	4	8	16	32	64
BM	134217728	131072	2048	256	128	128
CPP	134215680	126976	1024	0	0	0
AD	1024	132096	278656	348688	366244	549370
CP	1023	130047	259071	294143	224511	224511

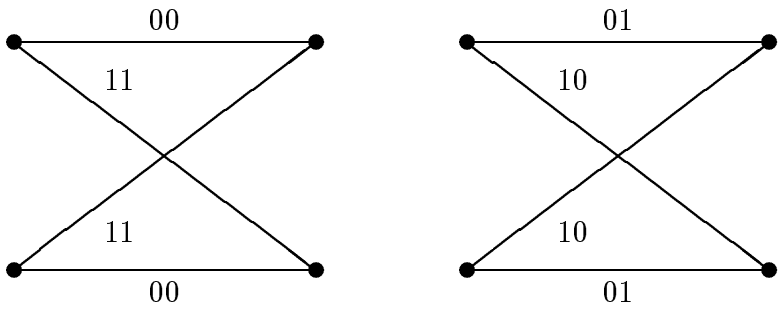
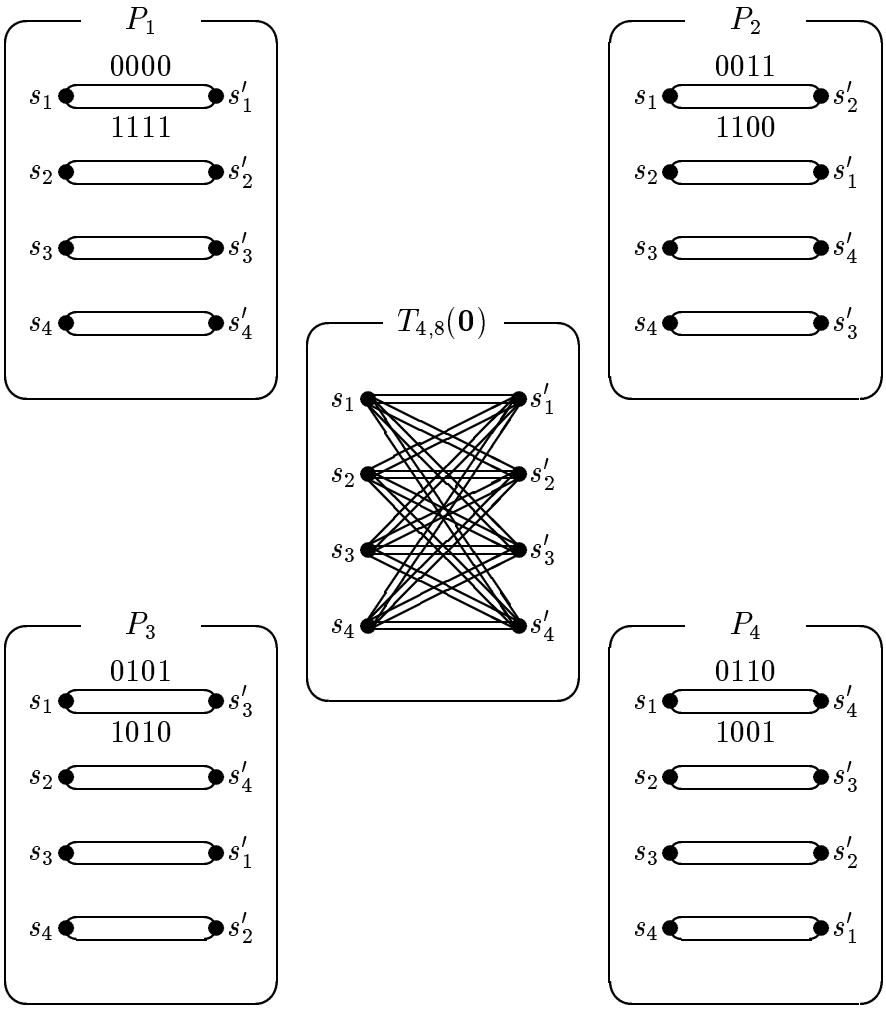


Figure 1: Two representative parallel sub-trellis diagrams in the fourth section of the 16-section minimal trellis diaram for $\mathrm{RM}_{5,2}$.

s	$\varphi_1(s)$	$\varphi_2(s)$	$\varphi_3(s)$	$\varphi_4(s)$
s_1	s'_1	s'_2	s'_3	s'_4
s_2	s'_2	s'_1	s'_4	s'_3
s_3	s'_3	s'_4	s'_1	s'_2
s_4	s'_4	s'_3	s'_2	s'_1

(a) One-to-one mapping $\varphi_i(s)$ with $1 \leq i \leq 4$.



(b) The detailed structure of $T_{4,8}(\mathbf{0})$.

Figure 2: The detailed structure of the parallel component $T_{4,8}(\mathbf{0})$ for $\mathrm{RM}_{4,2}$.