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Implicit Lagrangian for Generalized Complementarity Problems

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Abstract. This paper is concerned with an unconstrained optimization formulation of a generalized nonlinear complementarity problem (GNCP). This formulation is based on an extension of the implicit Lagrangian for the nonlinear complementarity problem (NCP) recently proposed by Mangasarian and Solodov. We show that the fundamental properties of the implicit Lagrangian for NCP are inherited to the case of GNCP. In particular, we investigate sufficient conditions under which any stationary point of the unconstrained optimization problem becomes a solution to the original GNCP.

1 Introduction

The nonlinear complementarity problem (NCP) [3, 10] is to find a vector $x \in R^n$ such that

$$[\text{NCP}] \qquad x \geq 0,\ F(x) \geq 0,\ x^T F(x) = 0,$$

where F is a mapping from R^n into R^n. In this paper, we consider the following generalization of NCP: Find $x \in R^n$ such that

$$[\text{GNCP}] \quad E(x) \geq 0,\ F(x) \geq 0,\ E(x)^T F(x) = 0,$$

where E is another mapping from R^n into R^n. We shall call this problem the generalized nonlinear complementarity problem (GNCP). As a generalization of NCP, the following problem has been studied in the literature under the name of the implicit complementarity problem [2, 4, 8, 9] or the quasi complementarity problem [6, 7]: Find $x \in R^n$ such that

$$[\text{QCP}] \quad x \geq m(x),\ F(x) \geq 0,\ (x - m(x))^T F(x) = 0,$$

where m is a mapping from R^n into R^n. Note that this problem can be interpreted as GNCP by substituting $E(x)$ for $x - m(x)$. But we prefer the former formulation GNCP because we shall treat the two mapping E and F in a symmetric manner.

The purpose of this paper is to study relationships between GNCP and the unconstrained optimization problem

$$\text{minimize } M_\alpha(x),$$

where $M_\alpha : R^n \to R^n$ is defined by

$$\begin{aligned} M_\alpha(x) &= E(x)^T F(x) + \frac{1}{2\alpha}\Big(\|(E(x) - \alpha F(x))_+\|^2 - \|E(x)\|^2 \\ &\quad + \|(F(x) - \alpha E(x))_+\|^2 - \|F(x)\|^2\Big), \end{aligned} \tag{1}$$

where $\|\cdot\|$ denotes the Euclidean norm and $(z)_+$ is the vector with components $\max\{0, z_i\}, i = 1, \ldots, n$. Notice that, by substituting x for $E(x)$, the function M_α reduces to the implicit Lagrangian considered by Mangasarian and Solodov [5] for NCP. So, the function M_α is a natural extension of the Mangasarian and Solodov's implicit Lagrangian. We show that the function M_α inherits the fundamental properties of the implicit Lagrangian for NCP. In particular, we investigate sufficient conditions under which any stationary point of M_α becomes a solution of GNCP. The latter results are generalizations of the one by Yamashita and Fukushima [11].

We mostly follow the notation used in [5]. If F is differentiable at x, then $\nabla F(x)$ denotes the $n \times n$ Jacobian matrix, with rows $\nabla F_i(x), i = 1, \ldots, n$, where $\nabla F_i(x)$ is the gradient vector $(\partial F_i(x)/\partial x_1, \ldots, \partial F_i(x)/\partial x_n)$. For any $J \subset \{1, \ldots, n\}$, $F_J(x)$ denotes the column vector with elements $F_i(x), i \in J$, and ∇F_J denotes the matrix with rows $\nabla F_i, i \in J$.

2 Main Results

It will be convenient to define the mappings $G : R^n \to R^n$ and $H : R^n \to R^n$ by

$$G(x) = (E(x) - \alpha F(x))_+ - E(x)$$

and

$$H(x) = (F(x) - \alpha E(x))_+ - F(x),$$

respectively.

Lemma 2.1 *The following three conditions are equivalent:*
(a) $E(x) \geq 0,\ F(x) \geq 0,\ E(x)^T F(x) = 0$;
(b) $G(x) = 0$;
(c) $H(x) = 0$.

Proof. Suppose (a) holds. Then for each i there are the following two cases:
Case 1: $E_i(x) = 0$ and $F_i(x) \geq 0$

$$G_i(x) = (-\alpha F_i(x))_+ = 0$$

and

$$H_i(x) = (F_i(x))_+ - F_i(x) = 0.$$

Case 2: $E_i(x) \geq 0$ and $F_i = 0$

$$G_i(x) = (E_i(x))_+ - E_i(x) = 0$$

and

$$H_i(x) = (-\alpha E_i(x))_+ = 0.$$

Hence, (b) and (c) hold. Next we show that (b) implies (a). Suppose (b) holds. For each i consider the following two cases:
Case 1: $E_i(x) - \alpha F_i(x) \geq 0$

$$G_i(x) = E_i(x) - \alpha F_i(x) + E_i(x) = -\alpha F_i(x) = 0.$$

This implies $F_i(x) = 0$, and hence $E_i(x) \geq 0$.
Case 2: $E_i(x) - \alpha F_i(x) < 0$

$$G_i(x) = E_i(x) = 0.$$

Hence, we have $E_i(x) = 0$ and $F_i(x) \geq 0$.
It follows from the two cases that (a) holds. We can show that (c) implies (a) in a similar way. □

The following theorem shows that a global minimizer of the function M_α defined by (1) is coincident with a solution of GNCP. This result is a natural extension of Theorem 2.1 in Mangasarian and Solodov [5].

Theorem 2.1 *For any $\alpha \in (1, \infty)$, $M_\alpha(x)$ is nonnegative for all $x \in R^n$ and $M_\alpha(x)$ vanishes if and only if x solves GNCP. That is, x is a global minimum of M_α if and only if x solves GNCP.*

Proof. For convenience, we represent $M_\alpha(x)$ as

$$M_\alpha(x) = \sum_{i=1}^{n} M_{\alpha,i}(x),$$

where

$$\begin{aligned} M_{\alpha,i}(x) &= E_i(x)^T F_i(x) + \frac{1}{2\alpha}((E_i(x) - \alpha F_i(x))_+^2 \\ &\quad -E_i(x)^2 + (F_i(x) - \alpha E_i(x))_+^2 - F_i(x)^2). \end{aligned}$$

Case 1: If $E_i(x) - \alpha F_i(x) \geq 0$ and $F_i(x) - \alpha E_i(x) \geq 0$, then

$$\begin{aligned} 2\alpha M_{\alpha,i}(x) &= -2\alpha E_i(x)F_i(x) + \alpha^2 F_i(x)^2 + \alpha^2 E_i(x)^2 \\ &= (E_i(x) - \alpha F_i(x))^2 + (\alpha^2 - 1)E_i(x)^2 \\ &\geq 0. \end{aligned}$$

Case 2: If $E_i(x) - \alpha F_i(x) < 0$ and $F_i(x) - \alpha E_i(x) \geq 0$, then

$$\begin{aligned} 2\alpha M_{\alpha,i}(x) &= -E_i(x)^2 + \alpha^2 E_i(x)^2 + F_i(x)^2 - F_i(x)^2 \\ &= (\alpha^2 - 1)E_i(x)^2 \geq 0. \end{aligned}$$

Case 3: If $E_i(x) - \alpha F_i(x) \geq 0$ and $F_i(x) - \alpha E_i(x) < 0$, then

$$\begin{aligned} 2\alpha M_{\alpha,i}(x) &= -F_i(x)^2 + \alpha^2 F_i(x)^2 + E_i(x)^2 - E_i(x)^2 \\ &= (\alpha^2 - 1)F_i(x)^2 \geq 0. \end{aligned}$$

Case 4: If $E_i(x) - \alpha F_i(x) < 0$ and $F_i(x) - \alpha E_i(x) < 0$, then

$$2\alpha M_{\alpha,i}(x) = 2\alpha E_i(x)F_i(x) - E_i(x)^2 - F_i(x)^2.$$

Moreover, we have the inequalities

$$E_i(x) < \alpha F_i(x) < \alpha^2 E_i(x)$$

and

$$F_i(x) < \alpha E_i(x) < \alpha^2 F_i(x).$$

Since $\alpha > 1$, these inequalities imply $E_i(x) > 0$ and $F_i(x)$. If $E_i(x) \geq F_i(x)$, then

$$2\alpha M_{\alpha,i}(x) > E_i(x)^2 - F_i(x)^2 \geq 0,$$

while, if $E_i(x) < F_i(x)$, then

$$2\alpha M_{\alpha,i}(x) > F_i(x)^2 - E_i(x)^2 > 0.$$

Consequently, we have $M_{\alpha,i}(x) \geq 0$.
Since these four cases exhaust all possibilities, it follows that $M_\alpha(x)$ is nonnegative for any x.

Next we show that $M_\alpha(x)$ vanishes if and only if x solves GNCP. If x solves GNCP, then $M_\alpha(x)$ vanishes by Lemma 2.1. We now show the only if part. Note that $M_\alpha(x) = 0$ implies $M_{\alpha,i}(x) = 0$ for all i. We consider the above four cases when $M_{\alpha,i}(x) = 0$.
Case 1: It follows that $E_i(x) = F_i(x) = 0$.
Case 2: It follows that $E_i(x) = 0, F_i(x) \geq 0$.
Case 3: It follows that $F_i(x) = 0, E_i(x) \geq 0$.
Case 4: This case is vacuous because $M_{\alpha,i}(x) > 0$ as shown in the first half of the proof.
Combining the outcomes of Cases 1, 2 and 3, we have that x solves GNCP. □

Note that the above results do not require the differentiability of the mappings E and F. If E and F are differentiable, then the function M_α is also differentiable and we have

$$\begin{aligned}
\alpha\nabla M_\alpha(x) &= \nabla E(x)^T[(E(x) - \alpha F(x))_+ - E(x) \\
&\quad -\alpha\left((F(x) - \alpha E(x))_+ - F(x)\right)] \\
&\quad +\nabla F(x)^T[(F(x) - \alpha E(x))_+ - F(x) \\
&\quad -\alpha\left((E(x) - \alpha F(x))_+ - E(x)\right)] \\
&= \nabla E(x)^T(G(x) - \alpha H(x)) \\
&\quad +\nabla F(x)^T(H(x) - \alpha G(x)). \qquad (2)
\end{aligned}$$

By Lemma 2.1, we immediately have the next result.

Theorem 2.2 *If E and F are differentiable at a solution $\bar{x}$ of GNCP, then $\nabla M_\alpha(\bar{x}) = 0$ for any $\alpha \in (1, \infty)$.*

Proof. If $\bar{x}$ is a solution of GNCP, then $G(\bar{x}) = H(\bar{x}) = 0$ by Lemma 2.1. Hence, we have $\nabla M_\alpha(\bar{x}) = 0$ by (2). $\square$

We establish the local uniqueness of global minima of $M_\alpha(x)$ under some conditions. This result is also a natural extension of Theorem 2.3 in [5].

Theorem 2.3 *Let E and F be twice differentiable at $\bar{x}$. If a solution $\bar{x}$ of GNCP satisfies strict complementarity condition $E(\bar{x}) + F(\bar{x}) > 0$ and if the gradient vectors $\nabla E_j(\bar{x})$, $j \in J$, and $\nabla F_k(\bar{x})$, $k \in K$, are linearly independent, where $J = \{j | E_j(\bar{x}) = 0\}$ and $K = \{k | F_k(\bar{x}) = 0\}$, then $\bar{x}$ is a locally unique global minimum solution of $M_\alpha(x)$ for any $\alpha \in (1, \infty)$.*

Proof. To prove the theorem, it suffices to show that $\nabla^2 M_\alpha(\bar{x})$ is positive definite. Since (2) yields

$$\begin{aligned} \alpha \nabla M_\alpha(x) &= (-\alpha \nabla E(x)^T + \nabla F(x)^T) H(x) \\ &\quad + (-\alpha \nabla F(x)^T + \nabla E(x)^T) G(x), \end{aligned}$$

and since $G(\bar{x}) = H(\bar{x}) = 0$, we have

$$\begin{aligned} \alpha \nabla^2 M_\alpha(\bar{x}) &= (-\alpha \nabla E(\bar{x})^T + \nabla F(\bar{x})^T) \nabla H(\bar{x}) \\ &\quad + (-\alpha \nabla F(\bar{x})^T + \nabla E(\bar{x})^T) \nabla G(\bar{x}). \end{aligned}$$

Moreover, since the strict complementarity condition gives

$$\nabla G(\bar{x}) = \begin{bmatrix} -\alpha \nabla E_J(\bar{x}) + \nabla F_J(\bar{x}) \\ 0_K \end{bmatrix} - \nabla E(\bar{x})$$

and

$$\nabla H(\bar{x}) = \begin{bmatrix} 0_J \\ -\alpha \nabla F_K(\bar{x}) + \nabla E_K(\bar{x}) \end{bmatrix} - \nabla F(\bar{x}),$$

we can write

$$\begin{aligned} \alpha \nabla^2 M_\alpha(\bar{x}) &= (-\alpha \nabla E_J(\bar{x})^T + \nabla F_J(\bar{x})^T)(-\alpha \nabla E_J(\bar{x}) + \nabla F_J(\bar{x})) \\ &\quad + \alpha \nabla E(\bar{x})^T \nabla F(\bar{x}) - \nabla F(\bar{x})^T \nabla F(\bar{x}) \end{aligned}$$

$$
\begin{aligned}
& +(-\alpha\nabla F_K(\bar{x})^T + \nabla E_K(\bar{x})^T)(-\alpha\nabla F_K(\bar{x}) + \nabla E_K(\bar{x})) \\
& +\alpha\nabla F(\bar{x})^T\nabla E(\bar{x}) - \nabla E(\bar{x})^T\nabla E(\bar{x}) \\
= \;& \alpha^2\nabla E_J(\bar{x})^T\nabla E_J(\bar{x}) - \alpha\nabla E_J(\bar{x})^T\nabla F_J(\bar{x}) - \alpha\nabla F_J(\bar{x})^T\nabla E_J(\bar{x}) \\
& +\nabla F_J(\bar{x})^T\nabla F_J(\bar{x}) + \alpha\nabla E(\bar{x})^T\nabla F(\bar{x}) - \nabla F(\bar{x})^T\nabla F(\bar{x}) \\
& +\alpha^2\nabla F_K(\bar{x})^T\nabla F_K(\bar{x}) - \alpha\nabla F_K(\bar{x})^T\nabla E_K(\bar{x}) - \alpha\nabla E_K(\bar{x})^T\nabla F_K(\bar{x}) \\
& +\nabla E_K(\bar{x})^T\nabla E_K(\bar{x}) + \alpha\nabla F(\bar{x})^T\nabla E(\bar{x}) - \nabla E(\bar{x})^T\nabla E(\bar{x}) \\
= \;& (\alpha^2-1)\Big(\nabla E_J(\bar{x})^T\nabla E_J(\bar{x}) + \nabla F_K(\bar{x})^T\nabla F_K(\bar{x})\Big) \\
= \;& (\alpha^2-1)\Big(\nabla E_J(\bar{x})^T\ \nabla F_K(\bar{x})^T\Big)\begin{pmatrix} \nabla E_J(\bar{x}) \\ \nabla F_K(\bar{x}) \end{pmatrix}.
\end{aligned}
$$

Since $\alpha > 1$ and the rows of $\nabla E_J(\bar{x})$ and $\nabla F_K(\bar{x})$ are linearly independent, it follows that $\nabla^2 M_\alpha(\bar{x})$ is positive definite as desired. $\square$

The converse of Theorem 2.2 is not necessarily true. We shall investigate conditions under which the converse is true. To begin with, we establish an interesting property of the mappings G and H.

Lemma 2.2 *Let D be an arbitrary $n \times n$ positive definite diagonal matrix, then for any $\alpha > 0$ and any $x \in R^n$*

$$(G(x) - \alpha H(x))^T D(H(x) - \alpha G(x)) \geq 0.$$

Proof. Let d_i be the ith diagonal component of D. By assumption, $d_i > 0$ for all i. We consider the following four cases.

Case 1: If $-\alpha F_i(x) + E_i(x) \geq 0$ and $-\alpha E_i(x) + F_i(x) \geq 0$, then we have

$$
\begin{aligned}
-\alpha G_i(x) + H_i(x) &= \alpha^2 F_i(x) - \alpha E_i(x) \\
&= \alpha(\alpha F_i(x) - E_i(x)) \leq 0
\end{aligned}
$$

and

$$
\begin{aligned}
G_i(x) - \alpha H_i(x) &= -\alpha F_i(x) + \alpha^2 E_i(x) \\
&= -\alpha(F_i(x) - \alpha E_i(x)) \leq 0.
\end{aligned}
$$

Case 2: If $-\alpha F_i(x) + E_i(x) < 0$ and $-\alpha E_i(x) + F_i(x) \geq 0$, then we have

$$-\alpha G_i(x) + H_i(x) = -\alpha(-E_i(x)) - \alpha E_i(x) = 0.$$

Case 3: If $-\alpha F_i(x) + E_i(x) \geq 0$ and $-\alpha E_i(x) + F_i(x) < 0$, then we have

$$G_i(x) - \alpha H_i(x) = -\alpha F_i(x) + \alpha F_i(x) = 0.$$

Case 4: If $-\alpha F_i(x) + E_i(x) < 0$ and $-\alpha E_i(x) + F_i(x) < 0$, then we have

$$-\alpha G_i(x) + H_i(x) = \alpha E_i(x) - F_i(x) > 0$$

and

$$G_i(x) - \alpha H_i(x) = -E_i(x) + \alpha F_i(x) > 0.$$

Thus, we obtain

$$d_i(-\alpha G_i(x) + H_i(x))(G_i(x) - \alpha H_i(x)) \geq 0$$

for each i, and hence,

$$(-\alpha G(x) + H(x))^T D(G(x) - \alpha H(x)) \geq 0.$$

The proof is complete. □

Lemma 2.3 *Let $\bar{x}$ be a stationary point of M_α and $\alpha \neq 1$. If E and F are differentiable at $\bar{x}$ and the matrices $\nabla E(\bar{x})$ and $\nabla F(\bar{x})$ are nonsingular, then the following three conditions are equivalent:*
(a) $G(\bar{x}) - \alpha H(\bar{x}) = 0$;
(b) $H(\bar{x}) - \alpha G(\bar{x}) = 0$;
(c) $H(\bar{x}) = G(\bar{x}) = 0$.

Proof. By assumption, $\nabla M_\alpha(\bar{x}) = 0$ and the matrices $\nabla F(\bar{x})$ and $\nabla E(\bar{x})$ are nonsingular. We first show the equivalence between (a) and (b). If (a) holds, then (2) yields

$$\nabla F(\bar{x})^T(-\alpha G(\bar{x}) + H(\bar{x})) = 0,$$

which together with the nonsingularity of $\nabla F(\bar{x})$ implies (b). Conversely, if (b) holds, then (2) yields

$$\nabla E(\bar{x})^T(-\alpha H(\bar{x}) + G(\bar{x})) = 0,$$

which together with the nonsingularity of $\nabla E(\bar{x})$ implies (a). Since (c) obviously implies (a) and (b), it remains to prove that (c) is implied by (a) or (b). Suppose (a) holds. Then (b) also holds as shown above. Moreover, since $\alpha \neq 1$, (a) and (b) are simultaneously satisfied only if $G(\bar{x}) = 0$ and $H(\bar{x}) = 0$. This completes the proof. $\square$

The next theorem states conditions under which the converse of Theorem 2.2 is true.

Theorem 2.4 *Let E and F be differentiable and let $\nabla E(x)$ and $\nabla F(x)$ be nonsingular for any x. If there exist $n \times n$ matrices $S(x)$ and $T(x)$ which satisfy the following conditions, then any stationary point of M_α is a solution of GNCP for any $\alpha \in (1, \infty)$:*
(a) $S(x)\nabla E(x)^T$ and $T(x)\nabla F(x)^T$ are positive semidefinite;
(b) $S(x)\nabla E(x)^T$ or $T(x)\nabla F(x)^T$ is positive definite;
(c) $S(x)\nabla F(x)^T + \nabla E(x)T(x)^T$ is positive definite diagonal matrix.

Proof. Let $\nabla M_\alpha(\bar{x}) = 0$ and put

$$\begin{aligned} X &= G(\bar{x}) - \alpha H(\bar{x}) \\ Y &= H(\bar{x}) - \alpha G(\bar{x}). \end{aligned}$$

Premultiplying (2) by $(S(\bar{x})^T X + T(\bar{x})^T Y)^T$ yields

$$\begin{aligned} \alpha(S(\bar{x})^T X + T(\bar{x})^T Y)^T \nabla M_\alpha(\bar{x}) &= (S(\bar{x})^T X + T(\bar{x})^T Y)^T (\nabla E(\bar{x})^T X + \nabla F(\bar{x})^T Y) \\ &= X^T S(\bar{x}) \nabla E(\bar{x})^T X + Y^T T(\bar{x}) \nabla F(\bar{x})^T Y \\ &\qquad + X^T S(\bar{x}) \nabla F(\bar{x})^T Y + Y^T T(\bar{x}) \nabla E(\bar{x})^T X \\ &= X^T S(\bar{x}) \nabla E(\bar{x})^T X + Y^T T(\bar{x}) \nabla F(\bar{x})^T Y \\ &\qquad + X^T \left(S(\bar{x}) \nabla F(\bar{x})^T + \nabla E(\bar{x}) T(\bar{x})^T \right) Y \qquad (3) \\ &= 0. \end{aligned}$$

The first two terms in (3) are both nonnegative by condition (a), while the third term is also nonnegative by condition (c) and Lemma 2.2. Hence, each term in (3) must vanish. It follows

from conditions (b) that $X = 0$ or $Y = 0$. This implies $G(\bar{x}) = H(\bar{x}) = 0$ by Lemma 2.3. Consequently, $\bar{x}$ is a solution of GNCP by the Lemma 2.1. $\square$

This theorem immediately yields the following corollary.

Corollary 2.1 *Let E and F be differentiable and let $\nabla E(x)$ and $\nabla F(x)$ be positive definite for any x. Let $k, l \in R$, $k + l > 0$ and $kl \geq 0$. If $k\nabla F(x)^T + l\nabla E(x)$ is a positive definite diagonal matrix, then any stationary point of M_α is a solution of GNCP for any $\alpha \in (1, \infty)$.*

Proof. Put $S(x) = kI$ and $T(x) = lI$. Then, under the given assumptions, it is easy to see that conditions (a), (b) and (c) in Theorem 2.4 are satisfied. $\square$

The next corollary is an extension of Theorem 2.2 in Yamashita and Fukushima [11].

Corollary 2.2 *Let E and F be differentiable and let ∇E be nonsingular for any x. If $\nabla F(x)\nabla E(x)^{-1}$ is positive definite, then any stationary point of M_α is a solution of GNCP for any $\alpha \in (1, \infty)$.*

Proof. Put $S(x) = 0$ and $T(x) = (\nabla E(x)^{-1})^T$. Then, it is straightforward to see that conditions (a), (b) and (c) in Theorem 2.4 are satisfied. $\square$

We obtain the following result as an immediate consequence of Corollary 2.2.

Corollary 2.3 *Let E and F be differentiable. If $\nabla E(x)$ and $\nabla F(x)$ are positive definite symmetric matrices, then a stationary point of M_α is a solution of GNCP for any $\alpha \in (1, \infty)$.*

Proof. We shall prove that the product of two positive definite symmetric matrices A and B is positive definite. Note that B may be represented as

$$B = B^{\frac{1}{2}} B^{\frac{1}{2}},$$

where $B^{\frac{1}{2}}$ is a symmetric matrix. Since the eigenvalues of AB are the same as those of $B^{\frac{1}{2}} A B^{\frac{1}{2}}$, it suffices to show that $B^{\frac{1}{2}} A B^{\frac{1}{2}}$ is positive definite. For any $x \in R^n$ such that $x \neq 0$, we have

$$x^T B^{\frac{1}{2}} A B^{\frac{1}{2}} x = (B^{\frac{1}{2}} x)^T A B^{\frac{1}{2}} x > 0,$$

since A is positive definite. Thus AB is positive definite. $\square$

It may be worth noting that the assumption that $\nabla E(x)$ and $\nabla F(x)$ are positive definite symmetric matrices cannot be replaced by the weaker assumption that $\nabla E(x)$ and $\nabla F(x)$ are positive definite and $\nabla E(x)$ or $\nabla F(x)$ is symmetric matrix in Corollary 2.3. In fact, consider the simple example with $n = 2$, where

$$E(x) = \begin{bmatrix} 2x_1 - x_2 + \frac{1}{3} \\ -x_1 + x_2 \end{bmatrix}$$

and

$$F(x) = \begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + x_2 + \frac{1}{3} \end{bmatrix}.$$

Note that $\nabla E(x)$ and $\nabla F(x)$ are positive definite and $\nabla E(x)$ is symmetric. For $\alpha = 2$, $\nabla M_\alpha(x) = 0$ holds at $x = (1, 2)^T$. But, this is not a solution of GNCP.

3 Reformulation of GNCP as NCP.

If E is invertible, then GNCP can be reduced to the following NCP: Find $z \in R^n$ such that

$$\text{[RNCP]} \qquad z^T Q(z) = 0,\ z \geq 0,\ Q(z) \geq 0,$$

where $z = E(x)$ and $Q(z) = F \circ E^{-1}(z)$. We consider the following implicit Lagrangian for RNCP.

$$\begin{aligned} N_\alpha(z) &= z^T Q(z) + \frac{1}{2\alpha}\left(\|(z - \alpha Q(z))_+\|^2 - \|z\|^2\right. \\ &\quad \left. + \|(Q(z) - \alpha z)_+\|^2 - \|Q(z)\|^2\right). \end{aligned}$$

Naturally, $N_\alpha(x)$ is also nonnegative for all x and vanishes at x if and only if x is a solution of RNCP. Moreover, If E and F are differentiable, then Q is also differentiable and we have

$$\nabla Q(z) = \nabla F(E^{-1}(z)) \nabla E^{-1}(z).$$

As we showed in [11], if $\nabla Q(x)$ is positive definite, then any stationary point of N_α is a solution of this NCP. So, we may solve GNCP by minimizing N_α if $\nabla Q(x)$ is positive definite. We investigate the relationship between N_α and M_α.

Lemma 3.1 *If E is differentiable and invertible, then*

$$\nabla E^{-1}(E(x)) = \nabla E(x)^{-1}.$$

Proof. Note that

$$\begin{aligned} I &= \nabla x \\ &= \nabla(E^{-1} \circ E)(x) \\ &= \nabla E^{-1}(E(x))\nabla E(x). \end{aligned}$$

Hence, $\nabla E^{-1}(E(x)) = \nabla E(x)^{-1}$. □

Lemma 3.2 *Let E and F be differentiable and let $\nabla E(x)$ be nonsingular for all x. Then the following two conditions are equivalent:*
(a) $\nabla Q(z)$ is positive definite for any z;
(b) $\nabla F(x)\nabla E(x)^{-1}$ is positive definite for any x.

Proof. If $\nabla Q(z)$ is positive definite, then $\nabla Q(E(x))$ is positive definite. By Lemma 3.1, we obtain

$$\begin{aligned} \nabla Q(E(x)) &= \nabla F(E^{-1}(E(x))\nabla E^{-1}(E(x)) \\ &= \nabla F(x)\nabla E^{-1}(E(x)) \\ &= \nabla F(x)\nabla E(x)^{-1}. \end{aligned}$$

Thus, $\nabla F(x)\nabla E(x)^{-1}$ is positive definite. We can show that the converse is also true in a similar manner. □

By this lemma, we obtain the next result.

Theorem 3.5 *Let E and F be differentiable and let $\nabla E(x)$ be nonsingular for all x. If $\nabla Q(z)$ is positive definite for any z, then any stationary point of M_α is a solution of GNCP. Moreover, if $\nabla F(x)\nabla E(x)^{-1}$ is positive definite for any x, then any stationary point of N_α is a solution of RNCP.*

Proof. Evident from Corollary 2.2 and Lemma 3.2. □

This theorem suggests that we need not reduce GNCP to RNCP by defining the implicit Lagrangian N_α. Rather it would be more convenient to deal with the implicit Lagrangian M_α which is defined directly for the original GNCP. Not to mention, if F is invertible, we can argue the same contents as above.

4 Conclusion

The generalized nonlinear complementarity problem GNLP has been reformulated as an unconstrained minimization problem by extending the implicit Lagrangian originally proposed for NCP. We have shown that the fundamental properties of the implicit Lagrangian for NCP are inherited to the case of GNCP. In particular, we have shown that, under some conditions, a stationary point of the extended implicit Lagrangian M_α is a solution of GNCP.

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