A Model of Time-Varying Flows on a Congested Multidestination Network

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Abstract

In designing an improved highway network, it is useful to predict traffic congestion under various settings. In particular, a dynamic flow model is suitable for a highly congested road network during the morning rush-hour. In the literature on dynamic traffic assignment models, however, attention has been paid for the most part to a single-destination network so far. The purpose of this paper is to propose a multicommodity dynamic flow model for predicting traffic congestion on a multidestination network. The model is formulated as an equilibrium programming problem, of which solution represents a 'commodity optimizing' flow pattern in the network. The problem consists of a set of convex programming subproblems and each subproblem amounts to minimizing the total cost for the users having a common destination. Existence of an equilibrium flow pattern is established under appropriate conditions.

Key Words Traffic assignment, time-varying flows, congested network, commodity optimizing flow, equilibrium programming.

1 Introduction

Highways leading into the downtown areas are highly congested during the morning rush-hour on every weekday. To design an improved highway network, it is helpful to predict a flow pattern on the congested road network over time under various hypothetical settings.

In the literature on transportation science, the following two concepts have been used to model traffic assignment, particularly for a static network.

- The *system-optimizing* flow pattern, which is realized when the network users cooperate so as to minimize the total cost of the system.
- The *user-optimizing* flow pattern, which is realized when the network users behave so as to minimize their individual costs.

Generally, we may consider the user-optimizing flow pattern to be more realistic for the road traffic, because then no user would have any incentive to change his/her route unilaterally.

A model of optimizing multiperiod flows on a congested network was first proposed by Merchant and Nemhauser (1978a, 1978b) and further investigated by Carey (1986, 1987, 1990), Carey and Srinivasan (1988), Ho (1980) and Tsujino, Fukushima and Ibaraki (1992). They assume that the business offices concentrate in a small area, so that the flows in the network have a single destination in common. In Merchant-Nemhauser (M-N) model, congestion effect is represented in terms of the exit capacity function associated with each arc, which controls outflow depending on the traffic volume on that arc. More precisely, the exit capacity function is determined in such a way that, as congestion increases, the outflow from arc per period increases at a lower rate, thereby giving rise to traffic delay. Note that M-N model deals with the system-optimizing flows.

Friesz, Luque, Tobin and Wie (1989) proposed two continuous versions of the M-N model, which are formulated as continuous time optimal control problems. One is a system-optimizing flow model, the other a user-optimizing flow model.

The assumption of single destination, however, prevents the model from being applied in wider situations. Big cities have several downtowns and this fact makes it difficult to

estimate traffic congestion. In fact, the above-mentioned models are not applicable to such cases, because they assume a single destination network. It is therefore necessary to formulate a multidestination network model that is useful in predicting dynamic traffic flows in a big city. Unfortunately few studies have been done on this subject so far. One of the few exceptions is the work by Wie, Friesz and Tobin (1990), who extented the model of Friesz et al. (1989) to a multidestination case. It is noted that the model of Friesz et al. assumes that arc exit functions are linear and separable with respect to each commodity, where the flows destined for a common node in a network are regarded as the same commodity. This restrictive assumption has made the model remarkably tractable.

We here focus on constructing a dynamic flow model applicable to a traffic flow over time on a congested multidestination network. We assume that the flows of each commodity are determined so as to minimize its own cost. Under this assumption, the flows of each commodity may be characterized as a solution of a convex programming problem, which involves arc capacity constraints that depend on the flows of all commodities. However, since those capacity constraints make the model difficult to deal with, we adopt a penalty function technique to transfer them into the objective function. This technique enables us to treat the problem as a minimum cost flow problem comprising a feasible set of the (single commodity) uncapacitated network flow and a nonlinear cost function.

For this model, we introduce a new idea called the *commodity-optimizing* flow pattern, which partially adopts the concept of user-optimization. The commodity-optimizing flow pattern may be considered an equilibrium in the model in the sense that the flows of each commodity will not be changed as long as the flows of other commodities remain the same. We show that the commodity-optimizing flow model can be formulated as an *equilibrium programming* problem, which consists of a set of nonlinear minimum cost flow problems mentioned above. A solution of the equilibrium programming problem corresponds to Nash equilibrium in game theory, if the users having a common destination are regarded as a player in the game.

2 A Dynamic Flow Model

2.1 Dynamic Flow on a Congested Network

Consider a network $G = \{\mathcal{N}, \mathcal{A}, \mathcal{K}\}$ consisting of a set of nodes $\mathcal{N} = \{1, \ldots, i, \ldots, n\}$, a set of directed arcs $\mathcal{A} = \{1, \ldots, j, \ldots, m\}$ and a set of destinations $\mathcal{K} = \{1, \ldots, k, \ldots, l\} \subset \mathcal{N}$, i.e., $l \leq n$ (see Figure 1). We regard the flows destined for destination k as commodity k. Throughout this paper, we shall make use of i, j and k to denote a typical index of node, arc and commodity, respectively. Furthermore we let $OUT(i)$ denote the set of arcs whose initial node is i and $IN(i)$ denote the set of arcs whose terminal node is i. We suppose that the planning horizon is divided into equal time periods $t = 1, \ldots, T$, and denote $\mathcal{P} = \{t \mid t = 1, \ldots, T\}$.

Associated with each commodity k are the following variables:

$x_j^k(t)$: the volume on arc j at the beginning of time period $t \in \mathcal{P}$,

$y_j^k(t)$: the outflow from arc j during time period $t \in \mathcal{P}$,

$z_j^k(t)$: the inflow to arc j during time period $t \in \mathcal{P}$,

$u_i^k(t)$: the exogenous supply at node i during time period $t \in \mathcal{P}$.,

v_j^k : the volume remaining on arc j at the end of time period T,

$w^k(t)$: the volume arriving at destination k during time period $t \in \mathcal{P}$.

We let the vector $f^k = (x^k, y^k, z^k, u^k, v^k, w^k)$ represent the flow pattern of commodity $k \in \mathcal{K}$, where x^k is the vector with components $x_j^k(t), j \in \mathcal{A}, t \in \mathcal{P}$, and the same convention is used for other variables. Moreover, let $f = (f^1, \ldots, f^k, \ldots, f^l)$ denote the flow pattern of all commodities, which exhibits the state of the entire system.

The next data are given for the network:

F_j^k : an initial volume of commodity k on arc j.,

S_i^k : a fixed total supply of commodity k springing from node $i \in \mathcal{N} - \{k\}$ throughout the planning horizon,

B_j : a capacity of arc j, i.e., the upper bound for the total volume $\sum_{k\in\mathcal{N}} x_j^k(t)$ of all commodities on arc j at each time period t,

$r_j(x)$: a function which gives an upper bound on the ratio of outflow from arc j to arc volume, when the total volume on the arc is x., This is assumed to be a decreasing concave function such that $0 \le r_j(x) \le 1$ on the interval $[0, B_j]$ (see Figure 2).

Throughout the paper, we assume the following assumption: No user passes through any arc within one time period.

For each commodity k, dynamic flows on network G must satisfy the following constraints:

$$(\mathrm{C1})^k \quad x_j^k(1) = F_j^k, \ \ \forall j \in \mathcal{A}.$$

Constraints $(\mathrm{C1})^k$ correspond to the initial conditions on arc flows.

$$(\mathrm{C2})^k \quad x_j^k(t) = x_j^k(t-1) - y_j^k(t-1) + z_j^k(t-1), \ \ \forall t \in \mathcal{P} - \{1\}, \forall j \in \mathcal{A}.$$
$$(\mathrm{C3})^k \quad v_j^k = x_j^k(T) - y_j^k(T) + z_j^k(T), \ \ \forall j \in \mathcal{A}.$$

Constraints $(\mathrm{C2})^k$ and $(\mathrm{C3})^k$ ensure conservation of flows on arcs. In particular, $(\mathrm{C2})^k$ represent conditions for intermediate time periods and $(\mathrm{C3})^k$ are terminal conditions.

$$(\mathrm{C4})^k \quad \sum_{j\in IN(i)} y_j^k(t) + u_i^k(t) = \sum_{j\in OUT(i)} z_j^k(t), \ \ \forall t \in \mathcal{P}, \forall i \in \mathcal{N} - \{k\}.$$
$$(\mathrm{C5})^k \quad \sum_{j\in IN(k)} y_j^k(t) = w^k(t), \ \ \forall t \in \mathcal{P}.$$

Constraints $(\mathrm{C4})^k$ and $(\mathrm{C5})^k$ ensure conservation of flows at nodes. Namely, $(\mathrm{C4})^k$ represent conditions for all nodes except destination k and $(\mathrm{C5})^k$ are conditions for destination k.

$$(\mathrm{C6})^k \quad \sum_{t\in\mathcal{P}} u_i^k(t) = S_i^k, \ \ \forall i \in \mathcal{N} - \{k\}.$$

Constraints $(\mathrm{C6})^k$ imply that aggregate supply at each node except destination k is fixed and the supply S_i^k at node i is distributed over the time horizon $\mathcal{P} = \{t \mid t = 1, \ldots, T\}$.

$$(\mathrm{C7})^k \quad f^k \ge 0.$$
$$(\mathrm{C8})^k \quad s_j^k(f) \le 0, \ \ \forall t \in \mathcal{P}, \forall j \in \mathcal{A},$$

where $s_j^k(f)$ is defined by

$$s_j^k(f) = y_j^k(t) - x_j^k(t) \cdot r_j \left(x_j^k(t) + \sum_{k' \in \mathcal{K} - \{k\}} x_j^{k'}(t) \right).$$

Constraints $(\mathrm{C7})^k$ ensure nonnegativity of the variables, and constraints $(\mathrm{C8})^k$ ensure that for each commodity the ratio of outflow $y_j^k(t)$ to arc volume $x_j^k(t)$ depends only on the total volume $x = \sum_{k \in \mathcal{K}} x_j^k(t)$. Note that conditions $(\mathrm{C7})^k$ and $(\mathrm{C8})^k$ along with the above-mentioned assumption on the function r_j imply $0 \leq \sum_{k \in \mathcal{K}} x_j^k(t) \leq B_j$.

Constraints $(\mathrm{C1})^k$ – $(\mathrm{C6})^k$ and $(\mathrm{C7})^k$ correspond to the flow conservation conditions and non-negativity flow conditions on a time-space network, respectively (Carey (1990) and Tsujino, Fukushima and Ibaraki (1992)). These constraints are independent of the flows of commodities other than k. On the contrary, constraints $(\mathrm{C8})^k$ depend on the flows of commodities other than k. To be specific, we use following notations:

$$\hat{H}^k = \{f^k \mid f^k \text{satisfies constraints } (\mathrm{C1})^k - (\mathrm{C7})^k\},$$

$$H = \prod_{k \in \mathcal{K}} \hat{H}^k,$$

$$H^k(f^{\tilde{k}}) = \{f^k \in \hat{H}^k \mid (f^k, f^{\tilde{k}}) \text{ satisfies constraints } (\mathrm{C8})^k\},$$

where $f^{\tilde{k}} = (f^1, \ldots, f^{k-1}, f^{k+1}, \ldots, f^l)$ is the flow pattern of all commodities except k.

2.2 The Model

On network G, the following three types of costs are introduced for each commodity k.

$C_j^k(t)$: a travel cost, which is imposed on each user passing through arc j during time period t.

D_j^k : a terminal cost, which is imposed on each user remaining on arc j at the end of time period T.

$E^k(t)$: a cost associated with the arrival time, which is imposed on each user arriving at destination k during time period t.

For the flow pattern f^k of commodity k, we assume that the following cost is imposed:

$$p^k(f^k) = \sum_{t\in\mathcal{P}}\sum_{j\in\mathcal{A}} C_j^k(t)x_j^k(t) + \sum_{j\in\mathcal{A}} D_j^k v_j^k + \sum_{t\in\mathcal{P}} E^k(t)w^k(t).$$

The first term is the sum of the travel costs on arc j throughout the time horizon $\mathcal{P}$, the second the sum of the costs incurred by remaining on arc j at the end of time period T, and the third the sum of the arrival costs at k throughout the time horizon $\mathcal{P}$.

We assume that the flows of each commodity are determined by optimizing its own cost. That is, if the flows $f^{\tilde{k}}$ of the other commodities are given, the flows of commodity k are characterized as a solution of the problem

$$\begin{aligned} \mathrm{SP}^k: \quad & \min_{f^k} \quad p^k(f^k) \\ & \text{s.t.} \quad f^k \in H^k(f^{\tilde{k}}). \end{aligned}$$

This problem involves the arc capacity constraints $(\mathrm{C8})^k$ that depend on the flows of the other commodities. This fact makes the problem difficult to deal with. We, therefore, transfer constraints $(\mathrm{C8})^k$ into the objective function using the penalty function

$$q^k(f^k, f^{\tilde{k}}) = M\sum_{t\in\mathcal{P}}\sum_{j\in\mathcal{A}} \left[\max\left\{0, s_j^k(f^k, f^{\tilde{k}})\right\}\right]^2,$$

where M is a sufficiently large positive constant.

By making use of the penalty function technique, we can formulate the following uncapacitated nonlinear minimum cost flow problem as a surrogate for SP^k.

$$\begin{aligned} \overline{\mathrm{SP}}^k: \quad & \min_{f^k} \quad h^k(f^k, f^{\tilde{k}}) = p^k(f^k) + q^k(f^k, f^{\tilde{k}}) \\ & \text{s.t.} \quad f^k \in \hat{H}^k. \end{aligned}$$

Though the solution of problem $\overline{\mathrm{SP}}^k$ usually violates constraints $(\mathrm{C8})^k$, it is expected to be a good approximation to the solution of SP^k if M is chosen sufficiently large (Luenberger (1984)).

Note that the constraints of subproblems $\overline{\mathrm{SP}}^k$ are independent of each other. More specifically, $\hat{H}^k$ is the feasible set of the (single commodity) uncapacitated network flow problem associated with commodity k. Furthermore, it will be shown in the next section that the cost function $h^k(f^k, f^{\tilde{k}})$ is a convex function of f^k for each fixed $f^{\tilde{k}}$.

The next definition gives a new solution concept for dynamic traffic flows on a congested multidestination network.

Definition 2.1 *A flow pattern $f = (f^1, \ldots, f^k, \ldots, f^l) \in H$ is said to be a commodity-optimizing flow pattern if it satisfies the following conditions: For each commodity k and given $f^{\tilde{k}}$, f^k minimizes the cost function $h^k(f^k, f^{\tilde{k}})$ subject to the condition $f^k \in \hat{H}^k$.*

The commodity-optimizing flow pattern will be achieved if the users of each commodity cooperate in minimizing their total cost. Because the costs of all commodities are interdependent, the outcome may then be compared to Nash equilibrium in a noncooperative game in which each commodity is a player.

We can formulate the commodity-optimizing flow model as an equilibrium programming problem (Garcia and Zangwill (1981), Zangwill and Garcia (1981), also see Appendix), which is comprised of subproblems $\overline{\mathrm{SP}}^k$ defined above. Namely, a commodity-optimizing flow pattern $f = (f^1, \ldots, f^k, \ldots, f^l)$ is an equilibrium in the sense that for all k and given $f^{\tilde{k}}$, f^k is an optimal solution of subproblem $\overline{\mathrm{SP}}^k$. In the following, such a commodity-optimizing flow pattern will be referred to as an equilibrium of the commodity-optimizing flow model, or simply an equilibrium of the model.

3 Existence of Equilibrium

We now prove that the commodity-optimizing flow model has an equilibrium under appropriate assumptions.

Theorem 3.1 *Suppose that there is an $\hat{f}^k \in \hat{H}^k$ such that $\hat{f}^k > 0$ for all $k \in \mathcal{K}$. Then the commodity-optimizing flow model has an equilibrium.*

Proof. We show that the model satisfies the three conditions of Theorem A.1 in Appendix. (i) Since the function $p^k(f^k)$ is linear in f^k, the convexity of the objective function $h^k(f^k, f^{\tilde{k}})$ is ensured, provided that the function $s_j^k(f^k, f^{\tilde{k}})$ is convex in f^k for each fixed $f^{\tilde{k}}$. We, therefore, have only to prove the convexity of the function $s_j^k(f^k, f^{\tilde{k}})$.

Let $f^{\tilde{k}}$ be fixed and consider two flow patterns $(f^k, f^{\tilde{k}}) \in H$ and $(f'^k, f^{\tilde{k}}) \in H$. For any $\alpha \in [0, 1]$, by the definition of $s_j^k(f^k, f^{\tilde{k}})$, we have

$$\alpha s_j^k(f^k, f^{\tilde{k}}) + (1-\alpha) s_j^k(f'^k, f^{\tilde{k}}) - s_j^k\left(\alpha f^k + (1-\alpha) f'^k, f^{\tilde{k}}\right)$$

$$= \alpha x_j^k(t) \left\{ R_j\left(\alpha x_j^k(t) + (1-\alpha) x'^k_j(t)\right) - R_j\left(x_j^k(t)\right) \right\}$$

$$+ (1-\alpha) x'^k_j(t) \left\{ R_j\left(\alpha x_j^k(t) + (1-\alpha) x'^k_j(t)\right) - R_j\left(x'^k_j(t)\right) \right\},$$

where $R_j(x) = r_j(x + \sum_{k' \in \mathcal{K}-\{k\}} x_j^{k'}(t))$. Since r_j is assumed to be concave and decreasing, R_j is also a concave decreasing function. Therefore we have

$$R_j\left(\alpha x_j^k(t) + (1-\alpha) x'^k_j(t)\right) \geq \alpha R_j\left(x_j^k(t)\right) + (1-\alpha) R_j\left(x'^k_j(t)\right).$$

Using the above equation and the fact that $R_j(x)$ is decreasing, we obtain

$$\alpha s_j^k(f^k, f^{\tilde{k}}) + (1-\alpha) s_j^k(f'^k, f^{\tilde{k}}) - s_j^k\left(\alpha f^k + (1-\alpha) f'^k, f^{\tilde{k}}\right)$$

$$\geq \alpha(1-\alpha) \left\{ R_j\left(x'^k_j(t)\right) - R_j\left(x_j^k(t)\right) \right\} \left(x_j^k(t) - x'^k_j(t)\right) \geq 0,$$

which shows that the function $s_j^k(f^k, f^{\tilde{k}})$ is convex in f^k.

(ii) Clearly each element of f^k is nonnegative and bounded above by the total volume of commodity k, which is given by $\sum_{i \in \mathcal{N}} S_i^k + \sum_{j \in \mathcal{A}} F_j^k$. Therefore the set of feasible solutions is bounded.

(iii) Equality constraints $(C1)^k$ – $(C6)^k$ in subproblem $\overline{SP}^k$ correspond to the flow conservation equations on a time-space network associated with commodity k (Carey (1990) and Tsujino et al. (1992)). Since the flow conservation for the sink is redundant, that equation has already been ignored in constraints $(C1)^k$ – $(C6)^k$. Therefore the Jacobian matrix of the equality constraints $(C1)^k$ – $(C6)^k$ has full rank. Finally the existence of positive flow $f^k \in \hat{H}^k$ ensures the remaining condition. □

4 A System Optimizing Model

For comparison purposes, we here briefly describe a system-optimizing flow model on a congested multidestination network.

When the flow pattern on the network is $f = (f^1, \ldots, f^k, \ldots, f^l)$, we suppose that the cost $\sum_{k\in\mathcal{K}} p^k(f^k)$ is imposed on the entire system, where $p^k(f^k)$ are specified in Subsection 2.2. Therefore, the system-optimizing flow pattern $f = (f^1, \ldots, f^k, \ldots, f^l)$, which is realized when all users behave so as to minimize the total cost in the system, may be obtained as

$$\begin{aligned} \min_{f} \quad & \sum_{k\in\mathcal{K}} p^k(f^k) \\ \text{s.t.} \quad & f^k \in H^k(f^{\tilde{k}}), \ \ \forall k \in \mathcal{K}. \end{aligned}$$

This problem contains nonlinear capacity constraints $(\text{C8})^k$, which are usually difficult to deal with. As in Subsection 2.2, we may also apply a penalty function technique to obtain the following system-optimizing flow model:

$$\begin{aligned} \min_{f} \quad & \sum_{k\in\mathcal{K}} h^k(f) \\ \text{s.t.} \quad & f \in H, \end{aligned}$$

where $h^k(f)$ are as defined in Subsection 2.2. If M is chosen sufficiently large, this model is expected to give a good approximation to the system-optimizing flow pattern. We can treat this model as a nonlinear minimum cost multicommodity flow problem on a time-space network. It is noted, however, that the objective function is in general nonconvex in the whole variables.

5 Conclusion

In this paper, we have presented a new solution concept, called the commodity-optimizing flow pattern, for dynamic flows on a congested multidestination network. We have shown that it can be formulated as an equilibrium programming problem, and proved that the model has an equilibrium under appropriate conditions.

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A Equilibrium Programming

In this appendix we briefly describe a basic idea of equilibrium programming presented by Garcia and Zangwill (1981) and Zangwill and Garcia (1981).

We consider a system consisting of M components. Each component of the system has a set of actions it can take. Let the vector $x^i \in R^{N_i}$ indicate an action taken by component i and let $x = (x^1, \ldots, x^i, \ldots, x^M) \in R^N$, where $N = N_1 + \cdots + N_M$, denote actions of all components, which exhibits a state of the entire system.

Generally, the set of possible actions of component i will depend on the actions taken by the other components. To be specific, let $x^{\tilde{i}} \in R^{N-N_i}$ be the subvector of x such that

$$x^{\tilde{i}} = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^M).$$

Namely the vector $x^{\tilde{i}}$ represents the actions of all components except i. We suppose that each component i can choose its action x^i from a feasible set, which may depend on the actions $x^{\tilde{i}}$ of the other components. We suppose further that each component attempts to minimize its disutility function $f^i(x^i, x^{\tilde{i}})$, which depends not only on the action of component i itself but also on the actions of the other components.

Given $x^{\tilde{i}}$, the action of component i may therefore be determined as a solution of the following optimization problem:

$$\begin{array}{lll} \mathrm{EP}^i & \min\limits_{x^i} & f^i(x^i, x^{\tilde{i}}) \\ & \text{s.t.} & g^i(x^i, x^{\tilde{i}}) \leq 0 \\ & & h^i(x^i, x^{\tilde{i}}) = 0, \end{array}$$

where $f^i : R^N \to R, g^i : R^N \to R^{r_i}$ and $h^i : R^N \to R^{s_i}$ are given functions and r_i and s_i are nonnegative integers.

We call $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^M)$ an equilibrium if $\bar{x}^i$ solves EP^i with given $\bar{x}^{\tilde{i}}$ for all $i = 1, \ldots, M$. The equilibrium programming problem is the problem of finding such an equilibrium.

The next theorem gives a sufficient condition for existence of an equilibrium. The proof may be found in Garcia and Zangwill (1981) and Zangwill and Garcia (1981).

Theorem A.1 *Suppose that the following three conditions hold:*

(i) For all i, the functions f^i and g^i are convex in x^i and h^i is linear in x^i;

(ii) the constraint set $D = \{x \mid g^i(x) \leq 0, h^i(x) = 0, i = 1, \ldots, M\}$ is bounded;

(iii) there is an $\hat{x} = (\hat{x}^i, \hat{x}^{\bar{i}}) \in D$ such that for all i and any $x = (x^i, x^{\bar{i}}) \in D$ the Jacobian matrix $\nabla_{x^i} h^i(x)$ has full rank and

$$g^i(\hat{x}^i, x^{\bar{i}}) < 0, \qquad h^i(\hat{x}^i, x^{\bar{i}}) = 0.$$

Then the equilibrium programming problem has an equilibrium.