One-Sided Error Quantum Pushdown Automata with Classical Stack Operations

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Abstract— We show that there is a non-context-free language which quantum pushdown automata with classical stack operations can recognize with one-sided error.

Keywords: quantum pushdown automata, quantum computation model, context-free-language

1 Introduction

Quantum pushdown automata were first introduced in [2]. Then it was shown that unitary quantum pushdown automata can recognize every regular language and some non-context-free languages in [1]. Another quantum pushdown automaton model was introduced in [3], stack operations of which are defined as classical operations.

In this paper, we show that there is a language which quantum pushdown automata with classical stack operations can recognize with one-sided error. We also show that this language cannot be recognized by non-deterministic pushdown automata, that is, it is not a context-free language. This implies that it cannot be recognized by probabilistic pushdown automata with one-sided error, either.

2 Preliminaries

In this section, we define quantum pushdown automata with classical stack operations (QCPDA). A QCPDA has an input tape to which a quantum head is attached, and has a classical stack to which a classical stack top pointer is attached. A QCPDA has a quantum state control. The quantum state control reads the stack top symbol pointed by the classical stack top pointer and reads the input symbol pointed by the quantum head. A word to be pushed is determined solely by the result of an observation of the quantum state control. We define QCPDA's formally as follows.

Definition 1 Quantum Pushdown Automata with Classical Stack Operations

A Quantum Pushdown Automaton with Classical Stack Operations (QCPDA) is defined as the following 8-tuple:

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \sigma, Q_{\mathrm{acc}}, Q_{\mathrm{rej}}),$$

where Q is a set of states, Σ is a set of input symbols including the left and the right end-marker $\{¢, \$\}$ respectively, Γ is a set of stack symbols including the bottom symbol Z, δ is a quantum state transition function ($\delta : (Q \times \Sigma \times \Gamma \times Q \times \{0,1\}) \longrightarrow \mathbb{C}$), q_0 is an initial state, σ is a function by which words to be pushed are determined ($\sigma : Q \setminus (Q_{\mathrm{acc}} \cup Q_{\mathrm{rej}}) \longrightarrow \Gamma^*$), $Q_{\mathrm{acc}}(\subseteq Q)$ is a set of accepting states, and $Q_{\mathrm{rej}}(\subseteq Q)$ is a set of rejecting states, where $Q_{\mathrm{acc}} \cap Q_{\mathrm{rej}} = \emptyset$. □

$\delta(q, a, b, q', D) = \alpha$ means that the amplitude of the transition from q to q' moving the quantum head to D ($D = 1$ means 'right' and $D = 0$ means 'stay') is α when reading an input symbol a and a stack symbol b. A configuration of the quantum portion of QCPDA M is a pair (q, k), where k is the position of the quantum head and q is in Q. It is obvious that the number of configurations of the quantum portion is $n|Q|$, where n is an input length.

A superposition of configurations of the quantum portion of QCPDA M is any element of $l_2(Q \times \mathbb{Z}_n)$ of unit length, where $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. For each configuration, we define a column vector $|q, k\rangle$ as follows:

- $|q, k\rangle$ is an $n|Q| \times 1$ column vector.
- The row corresponding to (q, k) is 1, and the other rows are 0.

For an input word $\boldsymbol{x}$ (i.e., the string on the input tape between ¢ and \$) and a stack symbol a, we define a time evolution operator $U_a^{\boldsymbol{x}}$ as follows:

$$U_a^{\boldsymbol{x}}(|q, k\rangle) = \sum_{q' \in Q, D \in \{0,1\}} \delta(q, x(k), a, q', D) \, |q', k + D\rangle ,$$

where $x(k)$ is the k-th input symbol of the input $\boldsymbol{x}$. If $U_a^{\boldsymbol{x}}$ is unitary (for any $a \in \Gamma$ and for any input word $\boldsymbol{x}$), that is, $U_a^{\boldsymbol{x}} U_a^{\boldsymbol{x}\dagger} = U_a^{\boldsymbol{x}\dagger} U_a^{\boldsymbol{x}} = I$, where $U_a^{\boldsymbol{x}\dagger}$ is the transpose conjugate of $U_a^{\boldsymbol{x}}$, then we say that the corresponding QCPDA is well-formed. This means that the QCPDA is valid in terms of the quantum theory. We consider only well-formed QCPDA's.

We define the notion of "words accepted by a QCPDA" in the following. For that purpose, we first describe the way how the quantum portion and the classical stack of a QCPDA work.

Let the initial quantum state and the initial position of the head be q_0 and '0' respectively. We define as $|\psi_0\rangle = |q_0, 0\rangle$. We also define as follows:

$$\begin{aligned} E_w &= span\{|q,k\rangle \,|\sigma(q) = w\}, \\ E_{\mathrm{acc}} &= span\{|q,k\rangle \,|q \in Q_{\mathrm{acc}}\}, \\ E_{\mathrm{rej}} &= span\{|q,k\rangle \,|q \in Q_{\mathrm{rej}}\}. \end{aligned}$$

We define the observable $\mathcal{O}$ as $\mathcal{O} = \oplus_j E_j$, where j is $w \in \Gamma^*$, 'acc', or 'rej'. For notational simplicity, we define the outcome of an observation corresponding to E_j to be j.

The computation of the QCPDA proceeds as follows:

For an input word $\boldsymbol{x}$, the quantum portion works as follows:

(a) Let $|\psi_{i+1}\rangle = U_a^{\boldsymbol{x}} |\psi_i\rangle$, where a is the stack top symbol.

(b) $|\psi_{i+1}\rangle$ is observed with respect to the observable $\mathcal{O} = \oplus_j E_j$. Let $|\phi_j\rangle$ be the projection of $|\psi_{i+1}\rangle$ to E_j. Then each outcome j is obtained with probability $|\,|\phi_j\rangle\,|^2$. Note that this observation causes $|\psi_{i+1}\rangle$ to collapse to $\frac{1}{|\,|\phi_j\rangle|} |\phi_j\rangle$, where j is the obtained outcome. Then go to (c).

The classical stack works as follows:

(c) Let the outcome of the observation be j. If j is 'acc' ('rej', resp.) then it outputs 'accept' ('reject', resp.). Otherwise (j is a word in Γ^* in this case), the stack top symbol is popped, and the word j is pushed. Then, go to (a) and repeat.

We call the above (a), (b) and (c) 'one step' collectively.

For a language L, if there exists a constant ε ($0 < \varepsilon \leq 1$) that does not depend on inputs, a QCPDA accepts any word in L with probability greater than $1-\varepsilon$, and it rejects any word not in L with certainty, then we say that L is recognized by the QCPDA with one-sided error.

To check well-formedness, the following theorem is shown in [3].

Theorem 1 QCPDA M is well-formed if the quantum state transition function of M satisfies the following conditions for any $a \in \Gamma$:

$$\begin{aligned} &(1) \quad \forall q_1, q_2, b \\ &\sum_{q',D} \overline{\delta(q_1, b, a, q', D)}\delta(q_2, b, a, q', D) \\ &= \begin{cases} 1 & (q_1 = q_2) \\ 0 & (q_1 \neq q_2), \end{cases} \\ &(2) \quad \forall q_1, q_2, b_1, b_2 \\ &\sum_{q'} \overline{\delta(q_1, b_1, a, q', 0)}\delta(q_2, b_2, a, q', 1) = 0, \end{aligned}$$

where $\overline{\delta(\cdot,\cdot,\cdot)}$ is the conjugate of $\delta(\cdot,\cdot,\cdot)$. □

For simplicity, we will handle only a subclass of QCPDA's such that we can decompose the quantum state transition function into two functions: one for changing states and the other for moving the quantum head. For $a \in \Sigma$ and $b \in \Gamma$, we adopt a linear operator $V_{a,b} : l_2(Q) \longrightarrow l_2(Q)$ for changing states, and a function $\Delta : Q \longrightarrow \{0,1\}$ for moving the quantum head. We assume that direction of the movement of the head is determined solely by the state to which the current state makes a transition. Then the transition function δ we are concerned with is described as follows:

$$\delta(q, a, b, q', D) = \begin{cases} \langle q'|\, V_{a,b}\, |q\rangle & (\Delta(q') = D) \\ 0 & (\Delta(q') \neq D), \end{cases}$$

where $\langle q'|\, V_{a,b}\, |q\rangle$ is the coefficient of $|q'\rangle$ in $V_{a,b}\, |q\rangle$. The following theorem is also shown in [3].

Theorem 2 QCPDA M is well-formed if, for any $a \in \Sigma$ and $b \in \Gamma$, the linear operator $V_{a,b}$ satisfies the following condition:

$$\sum_{q'} \overline{\langle q'|\, V_{a,b}\, |q_1\rangle}\, \langle q'|\, V_{a,b}\, |q_2\rangle = \begin{cases} 1 & (q_1 = q_2) \\ 0 & (q_1 \neq q_2) \end{cases}$$

□

3 On the power of QCPDA's with one-sided error

In this section, we show that QCPDA's can recognize the following language L_1 with one-sided error while probabilistic pushdown automata cannot.

$$L_1 = \left\{ u\#v\natural w\flat x\%y \;\middle|\; \begin{array}{l} u,v,w,x,y \in \{a,b\}^*, \\ \neg\left(\begin{array}{l} |u| = |v| = |w| = |y| \\ \text{and } v = w \end{array}\right) \\ \text{and } \neg\left(\begin{array}{l} |u| = |x| \text{ and} \\ |v| = |w| = 0 \end{array}\right) \\ \text{and } (y = v^R \text{ or } y = w^R) \end{array} \right\}$$

Theorem 3 QCPDA's can recognize L_1 with one-sided error.

(Proof)

We show a QCPDA M that recognizes L_1. The outline of M is as follows: M rejects any input not of the form $u\#v\natural w\flat x\%y = \{a,b\}^*\#\{a,b\}^*\natural\{a,b\}^*\flat$ $\{a,b\}^*\%\{a,b\}^*$. M consists of three components M_1, M_2 and M_3. M_1 processes the substring $u\#v\natural w\flat x$ as follows:

1. M_1 reads u taking $2|u|$ steps with the stack unchanged,
2. pushes each symbol of v one by one,
3. and reads w and x taking $|w|+|x|$ steps with the stack unchanged.

M_2 processes the substring $u\#v\natural w\flat x$ as follows:

1. M_2 reads u and v taking $|u|+|v|$ steps with the stack unchanged,
2. pushes each symbol of w one by one,
3. and reads x taking $2|x|$ steps with the stack unchanged.

M_3 processes the substring y checking whether the string on the stack is y^R or not.

M_1 and M_2 process the substring $u\#v\natural w\flat x\%$ in parallel using superposition. Firstly, we consider the case that ($|u| = |v| = |w| = |x|$ and $v = w$) or ($|u| = |x|$ and $|v| = |w| = 0$). In this case, M_1 and M_2 push the same symbol at the same time, and also read '%' at the same time. Thus, when reading '%', M_1 and M_2 interfere with each other and move to the rejecting state with certainty. (See Fig. 1).

Secondly, we consider the case that $\neg(|u| = |v| = |w| = |x|$ and $v = w)$ and $\neg(|u| = |x|$ and $|v| = |w| = 0)$. In this case, the operations to the stack are different between M_1 and M_2 at some time. Then the observation, the result of which corresponds to the stack operation, causes the superposition to collapse to either M_1 or M_2. Thus M moves to the third component M_3 with probability 1/2 with v or w on the stack. Thus if $v = y^R$ or $w = y^R$, M accepts the input with probability 1/4.

We define M in detail in the following.

M has the following states:

- $q_{1,u,1,-}$, $q_{1,u,0,-}$, $q_{1,v,1,-}$, $q_{1,v,0,a}$, $q_{1,v,0,b}$, $q_{1,w,1,-}$, $q_{1,x,1,-}$.
- $q_{2,u,1,-}$, $q_{2,v,1,-}$, $q_{2,w,1,-}$, $q_{2,w,0,a}$, $q_{2,w,0,b}$, $q_{2,x,1,-}$, $q_{2,x,0,-}$.
- $q_{3,y,1,-}$, $q_{3,y,0,pop}$.
- q_0, q_{acc}, q_{rej1}, q_{rej2}, q_{rej3}.

q_{acc} is an accepting state. q_{rej1}, q_{rej2} and q_{rej3} are rejecting states. The index of $q_{i,z,d,c}$ means that the state is used to process a substring z in M_i, $\Delta(q_{i,z,d,c}) = d$, and c represents the value of

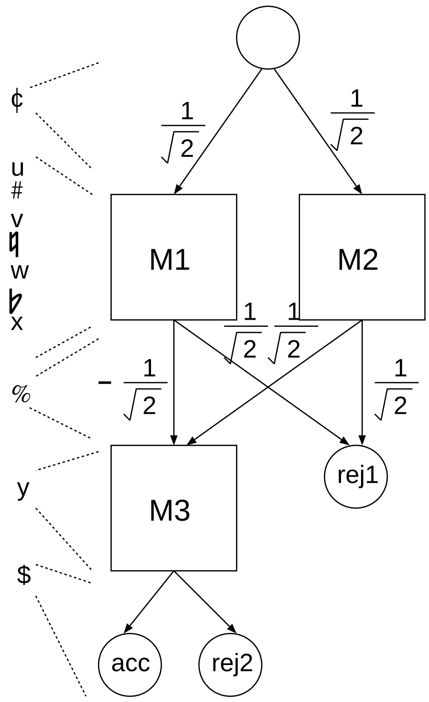

Figure 1: QCPDA M

$\sigma(q_{i,z,d,c})$ as follows: the stack top symbol is popped ($c =$ 'pop'), the stack is unchanged ($c = -$), and c is pushed (otherwise). The initial state is q_0.

The state transition diagrams of the components M_1, M_2 and M_3 are illustrated in Fig. 2. It is straightforward to see that the corresponding time evolution operators can be extended to be unitary by determining thus far undefined transitions properly by Theorem 2. □

We show that a non-deterministic pushdown automaton cannot recognize L_1. It is straight forward to see that this means that a probabilistic pushdown automaton cannot recognize L_1 with one-sided error, that is, L_1 is not a context-free language.

Theorem 4 A non-deterministic pushdown automaton cannot recognize L_1.

We use Ogden's lemma [4] to prove Theorem 4.

Ogden's Lemma

For any context-free language L, $\exists n \in \mathbb{N}$, the set of nonnegative integers, such that $\forall z \in L$, if d positions in z are "distinguished," with $d > n$, then $\exists u, v, w, x, y$ such that $z = uvwxy$ and

1. vx contains at least one distinguished position;
2. if r is the number of distinguished positions in vwx, then $r \leq n$;
3. $\forall i \in \mathbb{N}$, $uv^iwx^iy \in L$.

□

(Proof of Theorem 4)

Suppose that L_1 were a context-free language. We consider a word $z = p\#q\natural r\flat s\%t = a^{n!+n}\#a^n\natural a^{n!+n}\flat a^{n!+n}\%a^n \in L$. Let each symbol in q be distinguished. Then $z = uvwxy$ satisfies the conditions of Ogden's lemma. If either v or x contains a separator $(\#, \natural, \flat, \%)$, then $z' = uv^2wx^2y \notin L$. Note that vx contains at least one 'a' in q. Thus x must contains 'a's in t. (Otherwise $z' = uv^2wx^2y (= p'\#q'\natural r'\flat s'\%t') \notin L$ since $q' \neq t'^R$ and $r' \neq t'^R$). We assume that v contains c 'a's and $cd = n!$. Then $uv^{d+1}wx^{d+1}y (= p''\#q''\natural r''\flat s''\%t'') \notin L$ since $|p''| = |q''| = |r''| = |s''|$ and $q'' = r''$, a contradiction. Thus L_1 is not a context-free language. □

In [3], the following theorem is proved.

Theorem 5 QCPDA's can simulate any probabilistic pushdown automata with the same acceptance probability as that of the original probabilistic pushdown automata.

By Theorem 3, 4, and 5, we can conclude that one-sided error QCPDA's are strictly more powerful than one-sided error probabilistic pushdown automata.

4 Conclusion

We have shown that QCPDA's can recognize a language that is not context-free with one-sided error. Our conjecture is that L_1 cannot be recognized by probabilistic pushdown automata even with two-sided error, which implies that QCPDA's (with two-sided error) are strictly more powerful than probabilistic pushdown automata.

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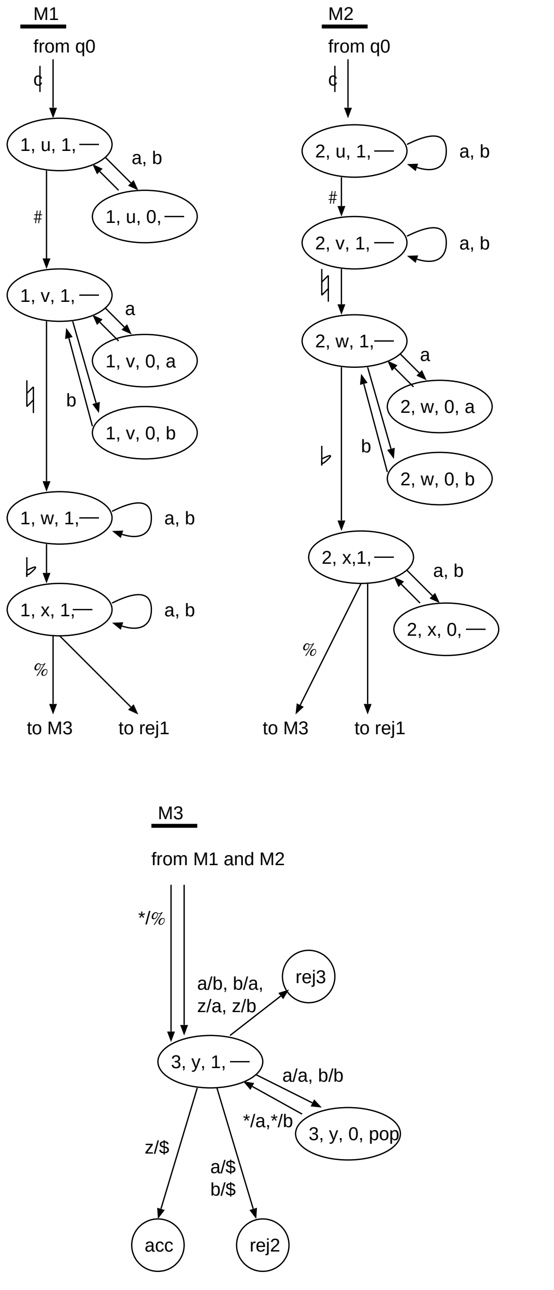

Figure 2: M_1, M_2 and M_3. Each edge label in M_1 and M_2 represents a corresponding input symbol, and each edge label in M_3 represents a corresponding pair of a stack top symbol and an input symbol.