An Analysis of Discrete-time Systems with a Single Delay: Razumikhin Theorem Approach

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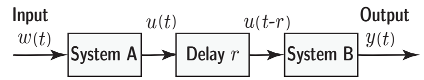
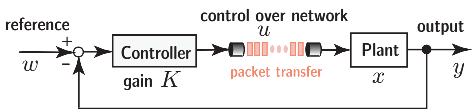
Motivation

- When we want to control something, robots, RC HELs and so on, under an ubiquitous networked circumstance, we can not neglect the effect of a delay that occurs over the network.



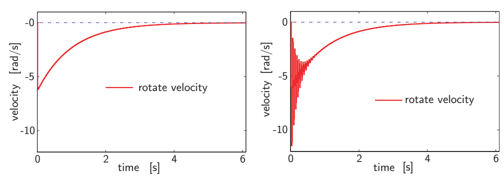
- Important to decide whether the existing control system with the delay works "safely" without a divergence.

Effects of the delay



- Plant model: Practical dc position servomechanism.
- Delay: Control u takes r steps to achieve the plant. At the current time t, the plant can not but use the delayed signal $u(t-r)$ instead of $u(t)$.
- Effects of the delay, $r = 1$, by simulation. (Since sampling time 0.01 [s], delay time is 0.01×1 [s].)

Effects of the delay



No delay   Delay $r = 1$ (0.01[s])

- The slight time-delay may effect on the behavior of the plant. Stability problem under the delay is important.

Our goal

- Show two stability criterion: delay-independent and delay-dependent stability criterion.
- Check the stability with numerical examples.

To achieve such the goal

1. Systems with a single delay, using an example.
2. Basic concept(definition) that "a systems is stable".
3. Delay-independent criteria and Delay-dependent criteria.
4. Numerical check.

Discrete-time system with a single delay

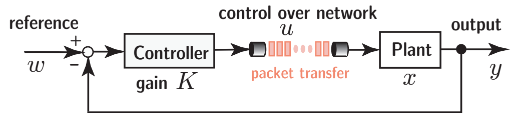
Given initial state: $x(\theta) \in \Re^n$, $\theta \in \{0, -1, \cdots, -r\}$,

$$x(t+1) = A_0 x(t) + A_1 x(t-r).$$

Example) Transitions of state x in case of $r = 2$:

$$\begin{aligned} x(1) &= A_0 x(0) + A_1 x(-2), \\ x(2) &= A_0 x(1) + A_1 x(-1), \\ &= A_0^2 x(0) + A_1 x(-1) + A_0 A_1 x(-2), \\ &\vdots \\ x(t) &= \mathsf{A}_0^{\mathsf{t,r}} x(0) + \mathsf{A}_1^{\mathsf{t,r}} x(-1) + \mathsf{A}_2^{\mathsf{t,r}} x(-2). \end{aligned}$$

Example of time-delay systems



Controller: $u(t) = K\{w(t) - y(t)\}$, Delay steps: r

Plant: $x(t+1) = Ax(t) + Bu(t-r)$, $y(t) = Cx(t)$

Closed-loop system with control over networks:

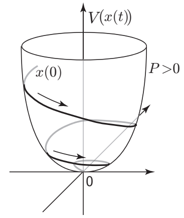
$$x(t+1) = Ax(t) - BKCx(t-r) + BKw(t-r)$$
$$y(t) = Cx(t)$$

Basic concept of stability(Lyapunov approach)

Lyapunov function:

$$V(x(t)) = x'(t)Px(t),$$

where $x \in \Re^n$, $P = P' > 0$. Distance between $x(t)$ and zeros.



As a step $t \to \infty$,
$x(t) \to 0$ means $V(x(t)) \to 0$.
$\Updownarrow$
$\Delta V = V(x(t+1)) - V(x(t)) < 0$.

If there exists P such that $\Delta V < 0$, then, the system is stable.

Stability condition of Razumikhin approach

- $\Delta V < 0$: the same of Lyapunov stability approach,
- Plus for some $p > 1$,

$$V(x(t+\theta)) < pV(x(t)) \text{ for all } \theta \in \{0, \cdots, -r\}.$$

Under the effect of the delay, a value of Lyapunov function V is decreasing to zero as $t \to \infty$. Then, difference ΔV has

$$\begin{aligned} \Delta V &= V(x(t+1)) - V(x(t)), \\ &< V(x(t+1)) - V(x(t)) \\ &\quad + \alpha\{px'(t)Px(t) - x'(t-r)Px(t-r)\}. \end{aligned}$$

Delay-independent stability criteria(Razumikhin apprch.)

The delay-system $x(t+1) = A_0 x(t) + A_1 x(t-r)$ is stable, if there exists a scalar

$$\alpha > 0$$

and a matrix $P = P' > 0$ such that

$$\begin{pmatrix} A_0' P A_0 - P + \alpha p P & A_0' P A_1 \\ A_1' P A_0 & A_1' P A_1 - \alpha P \end{pmatrix} < 0.$$

Remark:

- p : a parameter near 1 but more than 1.
- Fix α, result in the Linear Matrix Inequality form.

Delay-dependent stability criteria(Razumikhin apprch.)

- $\Delta V < 0$ plus for some $p > 1$,

$$V(x(t+\theta)) < pV(x(t)) \text{ for all } \theta \in \{0, \cdots, -r\}.$$

- Model transformation: $x(t+1) = A_0 x(t) + A_1 x(t-r)$

$$\Longrightarrow y(t+1) = \bar{A}_0 y(t) + \Sigma_{\theta=-2r}^{0} \bar{A}(\theta) y(t+\theta)$$

where

$$\begin{aligned} \bar{A}_0 &= A_1(I - A_1)^{-1} + I, \\ \bar{A}(\theta) &= A_1(I - A_1)^{-1}(I - A_0), \quad \theta \in \{-r, \cdots, 0\}, \\ \bar{A}(\theta) &= -A_1(I - A_1)^{-1} A_1, \quad \theta \in \{-2r, \cdots, -r\}. \end{aligned}$$

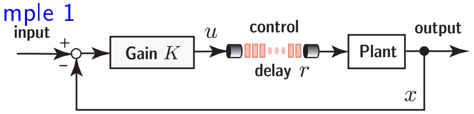
Delay-dependent stability criteria(Razumikhin apprch.)

The system $y(t+1) = \bar{A}_0 y(t) + \Sigma_{\theta=-2r}^{0} \bar{A}(\theta) y(t+\theta)$ is stable, if there exists scalars $\alpha_0 > 0$, $\alpha_1 > 0$ and matrices $P > 0$, R_0, R_1, such that

$$\bar{A}_0' P \bar{A}_0 - P + r(R_0 + R_1) < 0,$$
$$\Sigma_{\theta=-r}^{0} \bar{A}'(\theta) P \bar{A}(-r-\theta) < 0,$$
$$\begin{pmatrix} \alpha_k p P - R_k & A_0' P \bar{A}(\theta) \\ \bar{A}'(\theta) P \bar{A}_0 & \bar{A}'(\theta) P \bar{A}(\theta) - \alpha_k P \end{pmatrix} < 0, \quad k = 0, 1.$$

Remark: the transformed system is stable $\Longrightarrow$ the original time-delay system $x(t+1) = A_0 x(t) + A_1 x(t-r)$ is stable.

Example 1



Consider the following time-delay system with state $x = [x_1 \;\; x_2]'$:

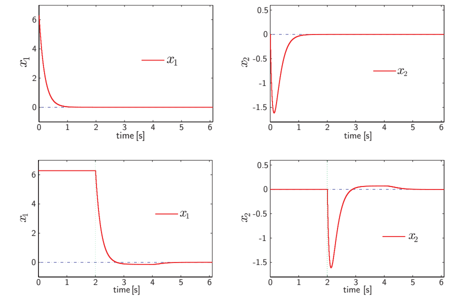
$$x(t+1) = \begin{bmatrix} 0.95 & 0 \\ -0.05 & 0.9 \end{bmatrix} x(t) + \begin{bmatrix} -0.0011 & 0.0001 \\ 0 & 0 \end{bmatrix} x(t-r),$$

where initial states $x(\theta) = [6.283 \;\; 0]'$ for all $\theta \in \{0, \cdots, -r\}$.

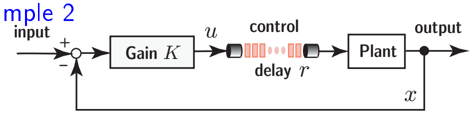
- In the delay-independent criteria, there exists the solutions of α and P.

Time-responses of Example 1

- Time-responses under the delay 0 [s] and 2.0 [s].



Example 2



Consider the following time-delay system with state $x = [x_1 \;\; x_2]'$:

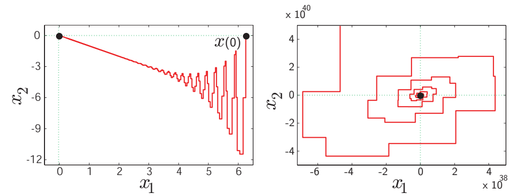
$$x(t+1) = \begin{bmatrix} 1.000 & 0.0095 \\ 0 & 0.9066 \end{bmatrix} x(t) + \begin{bmatrix} -0.049 & -0.044 \\ -0.9566 & -0.8733 \end{bmatrix} x(t-r),$$

where initial states $x(\theta) = [6.283 \;\; 0]'$ for all $\theta \in \{0, \cdots, -r\}$.

- In the delay-dependent criteria, there exists the solutions of α_0, α_1, R_0, R_1 and P under the delay 0.01 [s].

State transitions of Example 2

- State trasitions under the delay 0.01 [s] and 0.02 [s].



Delay 0.01 [s] Delay 0.02 [s]

- Under the delay 0.02 [s], the system is unstable.

Summary

- I showed two types of stability criterion for discrete-time systems with a single delay.
- I performed the stability check, illustrating time-responses and state transitions of example systems.

Future Plans

- Discrete-time ver. of another apprch to stability criterion.
- Time-delay systms with pointwise-in-time constraints.
- Model identification problem from experimental data.
- Control design, Experimental control and so on...